The Functional f(R) Approximation

泛函 f(R) 近似

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This chapter is a review of functional $f(R)$ approximations in the asymptotic safety approach to quantum gravity. It mostly focuses on a formulation that uses a non-adaptive cutoff, resulting in a second-order differential equation. This formulation is used as an example to give a detailed explanation for how asymptotic analysis and Sturm-Liouville analysis can be used to uncover some of its most important properties. In particular, if defined appropriately for all values $-\infty < R < \infty$, one can use these methods to establish that there are at most a discrete number of fixed points, that these support a finite number of relevant operators, and that the scaling dimension of high-dimension operators is universal up to parametric dependence inherited from the single-metric approximation. Formulations using adaptive cutoffs are also reviewed, and the main differences are highlighted.

本章综述了量子引力渐近安全方案中的泛函 $f(R)$ 近似。本章主要聚焦于采用非自适应截断的构造，该构造导出了一个二阶微分方程。本文以此构造为例，详细阐释了如何利用渐近分析和施图姆-刘维尔分析揭示其若干最重要的性质。特别地，若对所有取值 $-\infty < R < \infty$ 做恰当定义，便可通过这些方法证明: 该框架下最多存在离散个不动点，这些不动点仅支撑有限个相关算符，且高维算符的标度维数在单度量近似所继承的参数依赖范围内是普适的。本章也综述了采用自适应截断的构造，并 highlight 了二者的主要差异。

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关键词

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量子引力 · 重整化群 · 渐近安全 · $f(R)$ 近似——施图姆-刘维尔问题——渐近分析

Introduction

引言

One attempted route to a quantum theory of gravity is through the asymptotic safety program [1-4]. Although quantum gravity based on the Einstein-Hilbert action is plagued by ultraviolet infinities that are perturbatively non-renormalizable (implying the need for an infinite number of coupling constants), a sensible theory of quantum gravity might be recovered if there exists a suitable ultraviolet fixed point [1].

量子引力理论的一个尝试研究路径是渐近安全方案 [1-4]。尽管基于爱因斯坦-希尔伯特作用量的量子引力饱受微扰不可重整的紫外无穷大问题困扰 (这意味着需要无穷多个耦合常数)，但若存在合适的紫外不动点，我们仍可以得到一个自洽的量子引力理论 [1]。

The task is not just that of searching for an ultraviolet fixed point. They must also have the correct properties. Perturbatively renormalizable ones exist, for example, ”conformal gravity,” based on the square of the Weyl tensor, which thus corresponds to a Gaussian ultraviolet fixed point [5]. It is apparently not suitable however, because the theory is not unitary. Suitable unitary fixed points, if they exist, have to be non-perturbative. They must also satisfy phenomenological constraints, for example, they have to allow a renormalized trajectory with classical-like behavior in the infrared, since general relativity is confirmed by observation across many phenomena and to impressive precision. Of particular relevance for this chapter is that there should be a fixed point with a finite number of relevant directions (otherwise, it would be no more predictive than the perturbatively defined theory). Preferably, the theory should have only one fixed point or at least only a finite number (otherwise, again we lose predictivity).

我们的任务不只是寻找紫外不动点，找到的不动点还必须具备正确的性质。微扰可重整的不动点是存在的，例如基于外尔张量平方的“共形引力”对应一个高斯紫外不动点 [5]，但它显然并不适用，因为该理论不满足幺正性。合适的幺正不动点若存在，必然是非微扰的。它们还必须满足唯象约束: 例如必须允许存在一条在红外区表现出经典行为的重整化轨迹，因为广义相对论已经被众多观测现象以极高精度证实。对本章而言尤其重要的一点是，不动点必须只有有限个相关方向 (否则该理论的预言能力不会比微扰定义的理论更强)。理论最好只存在一个不动点，或至少只有有限个不动点，否则我们同样会失去预言能力。

Functional RG (renormalization group) equation [6-11] studies, first introduced by Wilson and Wegner many years ago [6, 7] (and called by them the ”exact RG”), have flourished into a powerful approach for investigating this possibility. These equations describe the flow of the Wilsonian effective action for some quantum field theory, under changes in an effective cutoff scale k . The asymptotic safety literature uses almost exclusively the flow equation for Γ_k which is, modulo minor details, the Legendre effective action (the generator of one-particle irreducible diagrams) cut off in the infrared by k . It was also formulated long ago [9] (in the sharp cutoff limit) and then rediscovered for smooth cutoffs much later in Refs. [10, 11]. Following Ref. [10], Γ_k is sometimes called the ”effective average action”; however, in this chapter, it will simply be called an effective action.

由威尔逊和韦格纳多年前首次提出的泛函 RG(重整化群) 方程研究 [6,7](二人将其称为“精确 RG”)，现已发展为研究这一可能性的强大方法。这些方程描述了量子场论中威尔逊有效作用量随有效截断标度 k 变化的流。渐近安全领域的研究几乎都使用 Γ_k 的流方程，除去细节不谈，它就是被 k 在红外区截断的勒让德有效作用量 (单粒子不可约图的生成泛函)。该框架也在很早之前就被构建出来 [9](在尖锐截断极限下)，之后很久又在文献 [10,11] 中被重新发现适用于光滑截断。按照文献 [10] 的说法，Γ_k 有时被称为“有效平均作用量”，但在本章中我们直接称之为有效作用量。

It is not practical to solve the full functional RG equations exactly. In a situation such as this, where there are no useful small parameters, one can only proceed by considering model approximations. These always proceed from the following observation: Wilsonian effective actions can be written as a sum over operators, where the coefficients are the couplings for these operators and they evolve with the scale k .

精确求解完整的泛函 RG 方程并不现实。在这种不存在有用小参数的情况下，我们只能通过模型近似来推进研究。所有近似都基于以下观察: 威尔逊有效作用量可以写为算符的求和形式，其中系数是对应算符的耦合，且耦合随标度 k 演化。

In fact, this sum should be restricted to local operators. This is the requirement of quasi-locality, which comes from the short-range nature of the Kadanoff blocking step in Wilsonian RG [6], when implemented in the continuum [12, 13]. A related point is that the Wilsonian RG is performed in Euclidean signature, so that ”short range” has a sensible meaning.

实际上，这个求和应当被限制在局域算符范围内。这是准局域性的要求，当威尔逊重整化群的卡达诺夫块步骤被推广到连续空间时 [12,13]，该要求就来源于这一步骤的短程性质 [6]。与之相关的一点是，威尔逊 RG 是在欧几里得符号下进行的，因此“短程”有明确的定义。

The problem is that for any general solution, this sum is infinite, over all possible local operators allowed by the symmetries (the space of all such couplings being known as ”theory space”). However, this motivates the simplest model approximation which is to truncate drastically the infinite-dimensional theory space to a handful of operators. An example is the original truncation studied by Reuter [2]:

问题在于，对于任意通解，这个求和是无穷级数，涵盖了对称性允许的所有可能局域算符 (所有这些耦合构成的空间被称为“理论空间”)。但这也引出了最简单的模型近似: 将无穷维的理论空间大幅截断为少量算符。罗伊特 [2] 最初研究的截断就是一个例子:

$$\Gamma_k\left[g_{\mu\nu}\right] = \int d^4x\sqrt{g}\left(u_0\left(k\right) + u_1\left(k\right)R\right), \tag{1}$$

which retains only the cosmological constant term and the scalar curvature R term. For obvious reasons, this is called the ”Einstein-Hilbert truncation.” Classically, $u_0 = -\lambda_{cc}/\left(8\pi G\right)$ and $u_1 = -1/\left(16\pi G\right)$, where λ_{cc} is the cosmological constant and G is Newton’s constant, but after quantum corrections, these couplings run with k in the functional RG. The minus sign in u_1 comes from working in Euclidean signature.

它仅保留了宇宙常数项和标量曲率 R 项。顾名思义，这被称为“爱因斯坦-希尔伯特截断”。经典层面上，$u_0 = -\lambda_{cc}/\left(8\pi G\right)$ 和 $u_1 = -1/\left(16\pi G\right)$ ，其中 λ_{cc} 是宇宙常数，G 是牛顿常数，但在量子修正后，这些耦合会随泛函 RG 中的 k 跑动。u_1 中的负号来源于欧几里得符号的约定。

Apart from RG symmetry, these truncations destroy pretty well all the properties that ought to hold. For example, scheme independence (i.e., independence on choice of cutoff or more generally universality) and modified BRST invariance [2, 14] (which encodes diffeomorphism invariance for the quantum field under the influence of the cutoff) cannot then be recovered. Furthermore, only by keeping an infinite number of local operators can the non-local long-range nature of (one-particle irreducible) Green’s functions be recovered (see, e.g., Ref. [15]). One has to trust that by considering ever less restrictive truncations, the description gets closer to the truth. There are some examples that go well beyond the Einstein-Hilbert truncation by keeping a large number of operators [16-19]. These are based around polynomial truncations, i.e., where everything is discarded except powers of some suitable local operators, typically the scalar curvature R again, up to some maximum degree. They appear to show convergence; in particular, the number of relevant operators is found to be 3.

除重整化群对称性外，这些截断几乎破坏了所有本应成立的性质。例如，方案独立性 (即对截断选择的独立性，更一般地说就是普适性) 和修正 BRST 不变性 [2, 14](它编码了受截断影响的量子场的微分同胚不变性) 都无法恢复。此外，只有保留无穷多个局域算符，才能得到 (单粒子不可约) 格林函数的非定域长程性质 (参见例如文献 [15])。人们必须相信，通过不断放宽截断限制，描述会越来越接近真实情况。目前已有一些超出爱因斯坦-希尔伯特截断的例子，这些例子保留了大量算符 [16-19]。它们基于多项式截断，即除了若干合适局域算符 (通常仍是标量曲率 R) 的幂次到某个最高次外，其余全部丢弃。这些结果似乎显示出收敛性；特别地，发现相关算符的数量为 3。

Another approximation in the asymptotic safety literature that is necessary in order to formulate diffeomorphism invariant truncations, such as Eq. (1), conflates the true (quantum) metric with the background metric. It is called the "single-metric" or "background field" approximation and will be described in the next section. It is harder to relax this approximation in any substantive way, although see Refs. [20-28] for some approaches.

渐近安全文献中，为构造微分同胚不变截断 (如式 (1)) 所需的另一近似将真实 (量子) 度量与背景度量混同了。这被称为“单度量”或“背景场”近似，我们将在下一节介绍。要实质性放松这一近似难度更大，不过相关研究方法可参见文献 [20-28]。

While very encouraging results are found from multiple studies of such finite-order truncations (see, e.g., the review [29]), successful implementations of more powerful approximations would build confidence in the scenario. The next step is to keep an infinite number of operators. Arguably, the simplest such truncation is to keep a full function $f(R)$, making the ansatz [22-24,30-43]

尽管对这类有限阶截断的多项研究都得到了非常令人鼓舞的结果 (参见例如综述 [29])，但若能成功实现更强大的近似，将会增强人们对该场景的信心。下一步是保留无穷多个算符。可以说，最简单的这类截断是保留一个完整函数 $f(R)$ ，据此构造的 ansatz 为 [22-24,30-43]

$$\Gamma_k[g] = \int d^4x\sqrt{g}f_k(R). \tag{2}$$

This is the functional $f(R)$ approximation which is the subject of this chapter. It is achieved by specializing to a maximally symmetric background manifold, either a four-sphere or four-hyperboloid.

这就是本章的主题——函数 $f(R)$ 近似。该近似通过限定在极大对称背景流形 (四维球面或四维双曲面) 上实现。

Closely related approximations have been studied in scalar-tensor [44,45] and unimodular [46] gravity and in three space-time dimensions [37]. In fact, the high-order finite-dimensional truncations [16-19] were developed by taking examples of these $f(R)$ equations and then further approximating to polynomial truncations.

与之密切相关的近似已在标量-张量引力 [44,45]、幺模引力 [46] 和三维时空 [37] 中得到研究。实际上，高阶有限维截断 [16-19] 正是通过选取这些 $f(R)$ 方程的例子，再进一步近似为多项式截断发展而来的。

Note that the functional $f(R)$ approximation actually goes beyond keeping a countably infinite number of couplings, the Taylor expansion coefficients $g_n = f^{(n)}(0)$, because a priori the large field parts of $f(R)$ contain degrees of freedom that are unrelated to all these g_n . For example, suppose that at large R , one finds that $f(R) \approx \exp\left(-a/R^2\right)$, where $a > 0$ is some parameter. Such an $f(R)$ is in the form of a standard counter-example in mathematical analysis. It has the property that $g_n = 0$ for all n .

请注意，函数 $f(R)$ 近似实际上超出了保留可数无穷多个耦合即泰勒展开系数 $g_n = f^{(n)}(0)$ 的范畴，因为先验来看，$f(R)$ 的大场部分包含了与所有这些 g_n 无关的自由度。例如，假设在大 R 处，得到 $f(R) \approx \exp\left(-a/R^2\right)$ ，其中 $a > 0$ 是某一参数。这样的 $f(R)$ 就是数学分析中标准反例的形式。它满足对所有 n 都有 $g_n = 0$ 的性质。

As Ref. [33] emphasized, the truncation (2) is as close as one can get to the local potential approximation (LPA) [47, 48], a successful approximation for scalar field theory in which only a general potential $V(\varphi)$ is kept for a scalar field φ (see, e.g., [47-52]). The LPA can be viewed as the start of a systematic derivative expansion [49], in which case this lowest order corresponds to regarding the field φ as constant. In rough analogy, an approximation of form (2) may be derived by working on a Euclidean signature space of maximal symmetry, where the scalar curvature R is constant. (Typically, a four-sphere is chosen.) In particular, techniques that have proved successful in scalar field theory [48-53] have been adapted to this very different context and used to gain substantial insight [43, 54-57].

正如文献 [33] 所强调的, 截断 (2) 已经是我们能得到的最接近局域势近似 (LPA)[47, 48] 的形式, 局域势近似是标量场论中一种成功的近似，它仅为标量场 φ 保留了一个一般势 $V(\varphi)$ (参见例如 [47-52])。LPA 可以被看作系统导数展开的起点，在该框架下，最低阶对应将场 φ 视为常数。大致类似地，形式 (2) 的近似可以在极大对称欧几里得签名空间上推导得到，其中标量曲率 R 是常数。(通常选取四维球面。) 特别地，在标量场论中已被证明有效的技术 [48-53] 已经被适配到这个完全不同的场景中，并被用来获得了大量深刻见解 [43, 54-57]。

The functional truncation (2) still has the problems that were highlighted earlier for its finite-dimensional counterparts. However, again one can hope that it is closer to the truth. One hint that this is in fact the case is covered at the end of this chapter. Assuming that the most recent version [43] does have a fixed-point solution, then it turns out that operators with high scaling dimension do begin to display universality - unfortunately up to an annoying parameter that remains which is clearly caused by the single-metric approximation.

泛函截断 (2) 依然存在我们之前在有限维对应情形中已经指出的问题。不过我们仍有理由认为它更接近真实情况。本章末尾会讨论一个能佐证这一点的线索。假设最新版本 [43] 确实存在不动点解，那么可以发现高标度维度的算符确实开始展现出普适性——遗憾的是，这种普适性仍受限于一个多余的参数，该参数显然是单度规近似带来的。

In this chapter, it will be explained how to construct functional $f(R)$ approximations and how to interpret them. Important properties of formulations that use an adaptive cutoff [22-24, 30-41] will be reviewed. Such cutoffs are called adaptive because they closely mimic the corresponding Hessian. They thus themselves depend on $f_k(R)$. In section "Cutoff Functions," we review the motivation for such cutoffs and details of their construction. The formulations using these cutoffs result in third-order differential equations with fixed singularities and problematic asymptotic behavior. Mostly, the chapter will focus on a non-adaptive cutoff formulation (as defined in section "Cutoff Functions") [42, 43]. This results in a second-order differential equation. We use it as an example to give a detailed exposition of the techniques, especially asymptotic analysis and Sturm-Liouville analysis, that can be used to prove properties of functional $f(R)$ approximations.

In particular, if the second-order formulation is taken to apply to only one of the two spaces (sphere or hyperboloid), the fixed-point solutions form a continuous set, and the eigenoperator spectrum is not quantized. However, if these spaces are joined together smoothly (through flat space at their boundary), these methods establish that there are at most a discrete number of fixed points and that the fixed points support a finite number of relevant operators and yield the result above for operators of high scaling dimension. They do not establish that such fixed points actually exist however. Such a demonstration requires more powerful numerical analysis and/or simpler fixed-point formulations [43].

本章将讲解如何构造泛函 $f(R)$ 近似并对其进行物理解释，同时回顾采用自适应截断 [22-24, 30-41] 的构造方案的重要性质。这类截断被称为自适应，是因为它可以很好地模拟对应黑塞矩阵的行为，因此它本身依赖于 $f_k(R)$ 。在“截断函数”一节中，我们会回顾这类截断的研究动机和构造细节。使用这类截断的构造方案会得到带有固定奇点且渐近行为存在问题的三阶微分方程。因此本章主要聚焦于非自适应截断方案 (定义见“截断函数”一节)[42, 43]，该方案得到的是二阶微分方程。我们将其作为实例，详细讲解可用于证明泛函 $f(R)$ 近似性质的技术，尤其是渐近分析和施图姆-刘维尔分析。具体而言，如果二阶公式仅适用于两个空间 (球面或双曲面) 中的一个，那么不动点解会构成一个连续集合，本征算符谱也不会量子化。但如果这些空间通过边界处的平坦空间光滑连接在一起，这些方法就能证明不动点最多只有离散个，且不动点对应有限个相关算符，并能得到上文关于高标度维度算符的结论。不过这些方法并不能证明这类不动点确实存在，要证明这一点需要更强大的数值分析和/或更简单的不动点构造方案 [43]。

Flow Equations

流方程

The starting point is the functional RG flow equation [10,11]:

出发点是泛函重整化群流方程 [10,11]:

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right], \tag{3}$$

where Str is a functional trace over space-time coordinates and indices that takes into account statistics of the fields. In momentum space, this is an integral over one-loop momentum. The right-hand side is a one-loop integral if Γ_k is taken to be classical. It includes all higher loops because Γ_k is actually given by the full (all orders) effective action. $\Gamma_k^{(2)}$ (Hessian) is the second variation of the effective action with respect to the fields, and $\mathcal{R}_k$ is an IR (infrared) cutoff for these fields. The cutoff scale k is related to RG ”time” t via $t = \ln(k/\mu)$, where μ is the standard arbitrary physical energy scale that appears in RG treatments (including in perturbative quantum field theory).

其中 STr 是对时空坐标和场指标的泛函迹，它包含了场的统计性质。在动量空间中，它对应单圈动量积分。若将 Γ_k 取为经典作用量，方程右侧就是一个单圈积分；但由于 Γ_k 实际上是全阶 (所有阶次) 有效作用量，因此该方程包含了所有高阶圈图贡献。$\Gamma_k^{(2)}$ (黑塞矩阵) 是有效作用量对场的二阶变分，$\mathcal{R}_k$ 是这些场的红外 (IR) 截断。截断标度 k 通过 $t = \ln(k/\mu)$ 与重整化群“时间”t 关联，其中 μ 是重整化群处理 (包括微扰量子场论中的处理) 中都会出现的标准任意物理能标。

An essential step in Wilsonian RG is to introduce dimensionless variables by multiplication of appropriate powers of the cutoff scale k . In the $f(R)$ approximation, the appropriate powers are just the canonical (a.k.a. engineering) ones:

威尔森重整化群的核心步骤是引入无量纲变量，方法是给各量乘上截断标度 k 的适当幂次。在 $f(R)$ 近似下，合适的幂次就是正则 (也称作工程) 维数对应的幂次:

$$\tilde{f}_k(\tilde{R}) \equiv \tilde{f}(\tilde{R}, t) = k^{-4} f_k(k^2 \tilde{R}), \ \tilde{R} = R/k^2. \tag{4}$$

In Wilsonian RG, one integrates out modes, starting with the high momentum modes first, by a coarse-graining procedure. Traditionally, after integrating out the modes, one has to rescale the action back to the original UV cutoff of the theory to see how the couplings change. By working with dimensionless quantities, this is taken care of automatically.

在威尔森重整化群中，我们通过粗粒化过程积掉模式，先积掉高动量模式。传统做法中，积掉模式后需要重新将作用量缩放回理论原本的紫外截断，才能看出耦合常数如何变化。使用无量纲量时，这一步会自动完成。

(From this point onward, it is convenient to drop the tilde denoting dimensionless quantities, unless otherwise specified, but the reader should assume that all the quantities are dimensionless.)

(从这里开始，除非另有说明，我们方便起见省略表示无量纲量的波浪号，读者默认所有量都是无量纲的即可。)

In this way, solutions to the flow equation will reveal all the fixed points of the theory, i.e., t -independent solutions $f(R,t) = f(R)$. Fixed points are characterized by the number of eigenoperators $v(R)$ (operators of definite scaling dimension) that flow into the fixed point when we increase the cutoff scale k . These are called relevant eigenoperators. Conversely irrelevant operators are the ones that flow away from the fixed point. The terms relevant (irrelevant) are common in the Wilsonian RG literature. Eigenoperators whose couplings do not flow (in some approximation or exactly) are called marginal. We do not need to discuss them in this chapter. Exceptionally, eigenoperators can appear that are ”redundant” or ”inessential” [1], corresponding to a change of variables in the theory [55,58,59].

通过这种方式，流方程的解会给出理论的所有不动点，也就是不依赖 t 的解 $f(R,t) = f(R)$ 。不动点的特征由特征算符 $v(R)$ (确定标度维数的算符) 的数量决定: 当我们增大截断标度 k 时，这些特征算符会流向不动点，被称为相关特征算符。反之，不相关算符是远离不动点的算符。相关/不相关这两个术语在威尔森重整化群文献中十分常用。耦合不流动 (在某些近似中或严格不流动) 的特征算符被称为边缘算符，我们在本章不需要讨论。特殊情况下，会出现“冗余”或“非本质”的特征算符 [1]，对应理论中的变量替换 [55,58,59]。

Eigenoperators are found by linearizing the flow equations around the fixed point and separating variables:

特征算符可以通过将流方程在不动点附近线性化、分离变量得到:

$$f_k(R) = f(R) + \varepsilon v(R)\, \mathrm{e}^{-\theta t} \tag{5}$$

where ε is a small parameter. This turns the flow equation into an eigenvalue problem where the RG eigenvalue θ is often called a ”critical exponent” in the asymptotic safety literature. From its associated $v(R)$, it can be similarly classified as relevant, irrelevant, marginal, or redundant. Thus, if $\mathrm{Re}\,\theta > 0$, then it is relevant, while if $\mathrm{Re}\,\theta < 0$, it is irrelevant. In statistical physics, non-redundant θ can be straightforwardly related to experimentally defined and measurable critical exponents; see, e.g., [60]. If computed correctly, an important property of a non-redundant θ is that it is universal, which means in particular that its value is independent of the regularization scheme and the choice of flow equation [58].

其中 ε 是小参数。这一步将流方程转化为一个特征值问题，重整化群本征值 θ 在渐近安全文献中通常被称为“临界指数”。根据对应的 $v(R)$ ，它也可以被分为相关、不相关、边缘或冗余四类: 当 $\mathrm{Re}\,\theta > 0$ 时，它是相关的；当 $\mathrm{Re}\,\theta < 0$ 时，它是不相关的。在统计物理中，非冗余的 θ 可以直接和实验定义、可测量的临界指数关联，参见例如文献 [60]。如果计算正确，非冗余的 θ 有一个重要性质: 它是普适的，这意味着它的数值不依赖于正规化方案和流方程的选择 [58]。

As already intimated, one is generally interested in those fixed points that have finitely many relevant operators, because their couplings become the free parameters in the theory, and will have to be fixed by experiments. Thus, theories based around these points are predictive and are safe from UV divergences when $k \to \infty$. The goal of the asymptotic safety program is to verify if such points exist for gravity, analyze their properties, and deduce their consequences, both qualitatively and quantitatively.

正如前文暗示，研究者通常关注具有有限个相关算符的不动点，因为这些算符对应的耦合就是理论中的自由参数，只需通过实验确定即可。因此，基于这类不动点建立的理论具有可预测性，并且当 $k \to \infty$ 时可以避免紫外发散。渐近安全纲领的目标就是验证引力中是否存在这类不动点，分析它们的性质，并推导其定性与定量结论。

Actually, the flow equation (3) requires a significant amount of adaptation to deal with the fact that quantum gravity is a gauge theory. In standard fashion, it therefore requires gauge fixing. This is commonly done by employing the background field method where the full (a.k.a. total) metric $\hat{g}_{\mu\nu}$ is split into a background $g_{\mu\nu}$ plus fluctuations (the quantum field):

实际上，由于量子引力是一种规范理论，流方程 (3) 需要做大量适配处理才能应对这一特性。按照标准方法，因此需要进行规范固定。这通常通过背景场方法实现: 该方法将完整 (也称为总) 度规 $\hat{g}_{\mu\nu}$ 分解为背景 $g_{\mu\nu}$ 加上涨落 (即量子场):

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{6}$$

In common with most of the literature, this chapter will only use a linear split, although other, non-perturbatively better motivated, splits are possible [29, 39]. Then the gauge fixing is imposed on the quantum field $h_{\mu\nu}$ in such a way that diffeomorphism invariance of the background metric $g_{\mu\nu}$ is retained:

与现有大多数文献一致，本章仅采用线性分解，尽管也可以使用其他非微扰意义下动机更充分的分解方式 [29, 39]。随后规范固定被施加在量子场 $h_{\mu\nu}$ 上，同时保留背景度规 $g_{\mu\nu}$ 的微分同胚不变性:

$$F_\mu = \nabla^\upsilon h_{\mu\upsilon} - \frac{1}{4}\nabla_\mu h^\upsilon_\upsilon, \tag{7}$$

where the covariant derivative and raised indices are defined using the background metric. The process of fixing a gauge adds the gauge fixing term

其中协变导数与升指标均通过背景度规定义。规范固定过程会添加规范固定项

$$\Gamma_{gf} = \frac{1}{2\alpha}\int d^4x\sqrt{g}g^{\mu\upsilon}F_\mu F_\upsilon \tag{8}$$

to the effective action and leads also to a ghost action. In practice, the Landau gauge is chosen: $\alpha \to 0$. Finally, it proves useful to make a change of variables, this is explained in (12), and this leads to further, auxiliary, fields.

到有效作用量中，同时还会产生鬼作用量。实际计算中通常选择朗道规范: $\alpha \to 0$ 。最后，改变量被证明是有用的，这一点在 (12) 中有解释，该操作还会引入额外的辅助场。

The true solution involves arbitrarily complicated interactions to arbitrarily high order between all these fields, with symmetries (in particular, background field diffeomorphism invariance and modified BRST invariance [14, 61]) making only partial improvements to this state of affairs. The next steps in the approximation drastically truncate all of this [2]. It can be summarized as follows. Only the one-loop contributions from the bilinear ghost and auxiliary field and fluctuation field actions are retained, i.e., on the right-hand side of the flow equation (3), only the Hessian from the classical action for these fields is used. The flow of the bit of the effective action that only depends on the background metric is therefore reproduced correctly at one loop. For the part beyond one loop, the correct Hessian in (3) for the metric,

真实解包含所有这些场之间任意高阶、任意复杂的相互作用，对称性 (尤其是背景场微分同胚不变性和修正 BRST 不变性 [14, 61]) 也只能部分改善这一状况。后续近似步骤会对所有这些内容做大幅截断 [2]，可以总结如下: 仅保留双线性鬼、辅助场和涨落场作用量的单圈贡献，也就是在流方程 (3) 的右侧，仅使用这些场经典作用量的黑塞矩阵。因此，仅依赖背景度规的那部分有效作用量的流在单圈阶可以被正确重现。对于单圈以上的部分，(3) 中关于度规的正确黑塞矩阵

$$\frac{\delta^2\Gamma_k}{\delta h_{\mu\nu}(x)\,\delta h_{\alpha\beta}(y)}, \tag{9}$$

is replaced by one in which the functional derivatives are with respect to the background field instead:

被替换为一个对背景场求泛函导数的形式:

$$\frac{\delta^2\Gamma_k}{\delta g_{\mu\nu}(x)\,\delta g_{\alpha\beta}(y)}. \tag{10}$$

This is the single-metric, or background field, approximation. It is almost always applied in asymptotic safety investigations. The review [28] covers exceptions. It should be emphasized that already at one loop the single-metric approximation is not correct, because the dependence of the effective action on $h_{\mu\upsilon}$ has no direct relation to its dependence on $g_{\mu\upsilon}$. The replacement above would be correct only if the effective action were a functional of the full metric (6) alone, but that relation is broken at the classical level by the gauge fixing term (8) (and corresponding ghost action). Nevertheless, the replacement is attractive as a model, because it leaves us with a flow equation for $\Gamma_k[g]$ that depends only on the background metric and in a diffeomorphism invariant way.

这就是单度量 (又称背景场) 近似。它几乎被应用于所有渐近安全相关研究中，综述文献 [28] 讨论了例外情况。需要强调的是，单度量近似即使在单圈阶也是不正确的，因为有效作用量对 $h_{\mu\upsilon}$ 的依赖和对 $g_{\mu\upsilon}$ 的依赖没有直接关联。只有当有效作用量仅是完整度规 (6) 的泛函时，上述替换才成立，但规范固定项 (8)(以及对应的鬼作用量) 在经典层面就破坏了这一关系。尽管如此，作为一种模型该替换仍有其吸引力，因为它最终得到的 $\Gamma_k[g]$ 的流方程仅依赖背景度规，且满足微分同胚不变性。

Now by choosing the background manifold to be one of maximal symmetry, all diffeomorphism invariants can be related to either the volume or the scalar curvature R, which is a constant: $\partial_\mu R = 0$. In this way, the effective action has been reduced to (2): the functional $f(R)$ approximation.

现在，如果选择背景流形为最大对称流形，所有微分同胚不变量都可以联系到体积或标量曲率 R，后者是一个常数: $\partial_\mu R = 0$。通过这种方式，有效作用量被约化为 (2) 式的形式: 这就是泛函 $f(R)$ 近似。

Plugging this with appropriately scaled fields (4) (and coordinates $\tilde{x}^\mu = kx^\mu$), into the flow equation (3), one readily derives the form of the left-hand side:

将经过恰当标度变换的场 (4)(以及坐标 $\tilde{x}^\mu = kx^\mu$) 代入流方程 (3)，我们可以很容易推导出方程左侧的形式:

$$\partial_t \Gamma_k = \int d^4x\sqrt{g}\left[\partial_t f_k(R) + 4f_k(R) - 2Rf_k'(R)\right]. \tag{11}$$

The right-hand side of the flow equation depends on the detailed way the quantum corrections are handled, which differs between authors [22-24, 30-43]. For this, we need to compute the second variation of Γ_k with respect to the fields. First, the gauge fixing term (8) is chosen, and the ghost action is derived. Then the transverse traceless (a.k.a. York) decomposition of the metric [62] is used:

流方程的右侧依赖于量子修正的具体处理方式，不同研究者的处理有所不同 [22-24, 30-43]。为此，我们需要计算 Γ_k 对场的二阶变分。首先选定规范固定项 (8)，推导出鬼作用量，再使用度规的横向无迹 (又称约克) 分解 [62]:

$$h_{\mu\upsilon} = h^T_{\mu\upsilon} + \nabla_\mu\xi_\upsilon + \nabla_\upsilon\xi_\mu + \nabla_\mu\nabla_\upsilon\sigma + \frac{1}{d}g_{\mu\upsilon}\bar{h}, \tag{12}$$

which separates physical degrees of freedom, viz., $h^T_{\mu\nu}$ and $\bar{h}$, from the unphysical ones associated with gauge degrees of freedom, namely, ξ_μ and σ. These fields satisfy

该分解将物理自由度，即 $h^T_{\mu\nu}$ 和 $\bar{h}$，与和规范自由度相关的非物理自由度，即 ξ_μ 和 σ 分离开来。这些场满足

$$h^{T\mu}{}_{\mu} = 0,\ \nabla^\mu h^T_{\mu\upsilon} = 0,\ \nabla^\mu\xi_\mu = 0,\ \bar{h} = h - \nabla^2\sigma. \tag{13}$$

Expressing $\sqrt{\hat{g}}$ and $\widehat{R}$, where the latter is the curvature of the full metric (6), to quadratic order in these fields, the elements of the Hessian can be determined for these components. For example, for the physical components, one finds

将 $\sqrt{\hat{g}}$ 和代表完整度规 (6) 曲率的 $\widehat{R}$ 按场展开到二次阶，就可以确定这些分量对应的黑塞矩阵元。例如，对于物理分量可以得到

$$\Gamma^{(2)}_{h^T_{\mu\nu}h^T_{\alpha\beta}} = -\frac{1}{2}\left[f_k'(R)\left(-\nabla^2+\frac{1}{6}R\right)+\left(f_k-\frac{1}{2}Rf_k'\right)\right]\delta^{\mu\nu,\alpha\beta}, \tag{14}$$

$$\Gamma^{(2)}_{hh} = \frac{1}{16}\left[9f_k''\left(-\nabla^2-\frac{R}{3}\right)^2+3f_k'\left(-\nabla^2-\frac{R}{3}\right)-\left(Rf_k'-2f_k\right)\right], \tag{15}$$

where the right-hand side is evaluated at $\hat{g}_{\mu\nu}=g_{\mu\nu}$, in preparation for the single-metric approximation. We can write these more compactly if we introduce

其中右侧在 $\hat{g}_{\mu\nu}=g_{\mu\nu}$ 处计算，为单度规近似做准备。引入以下符号后，我们可以将这些式子写得更简洁:

$$E_k(R) = 2f_k(R) - Rf_k'(R), \tag{16}$$

which is the equation of motion that follows from the action (2), and express them instead using the natural Laplacian Δ_s for a spin s component field (on a maximally symmetric background) [33]:

它是由作用量 (2) 导出的运动方程，我们改用 (最大对称背景上) 自旋 s 分量场的自然拉普拉斯算子 Δ_s 来重写这些式子 [33]:

$$\Delta_0 = -\nabla^2 - \frac{R}{3},\ \Delta_1 = -\nabla^2 - \frac{R}{4},\ \Delta_2 = -\nabla^2 + \frac{R}{6}. \tag{17}$$

A similar decomposition is applied to the ghost action. In the formulation of Ref. [33], the contributions to the Hessian coming from the gauge degrees of freedom from the metric and the ghosts cancel each other exactly. Finally, including the contributions of the auxiliary fields that encode the Jacobians due to the transverse traceless decomposition of the metric and the ghost fields gives the full flow equation (3) in the single-metric and functional $f(R)$ approximation:

对鬼作用量也做了类似分解。在文献 [33] 的表述中，度规规范自由度和鬼场对黑塞矩阵的贡献恰好相互抵消。最后，计入由度规横无迹分解和鬼场引入的雅可比行列式对应的辅助场贡献后，我们得到单度规泛函 $f(R)$ 近似下完整的流方程 (3):

$$V\left(\partial_t f_k(R) + 2E_k(R)\right) = \mathcal{T}_2 + \mathcal{T}_0^{\bar{h}} + \mathcal{T}_1^{\mathrm{Jac}} + \mathcal{T}_0^{\mathrm{Jac}}, \tag{18}$$

where $V=\int d^4x\sqrt{g}$ is the volume of the manifold and the $\mathcal{T}$ objects are the following space-time traces:

其中 $V=\int d^4x\sqrt{g}$ 是流形的体积，$\mathcal{T}$ 是如下时空迹:

$$\mathcal{T}_2 = \mathrm{Tr}\left[\frac{d_t \mathcal{R}_k^T}{-f_k'(R)\,\Delta_2 - E_k(R)/2 + 2\mathcal{R}_k^T}\right], \tag{19}$$

$$\mathcal{T}_0^{\bar{h}} = \mathrm{Tr}\left[\frac{8 d_t \mathcal{R}_k^{\bar{h}}}{9 f_k''(R)\,\Delta_0^2 + 3 f_k'(R)\,\Delta_0 + E_k(R) + 16\mathcal{R}_k^{\bar{h}}}\right], \tag{20}$$

$$\mathcal{T}_1^{\mathrm{Jac}} = -\frac{1}{2}\,\mathrm{Tr}\left[\frac{d_t \mathcal{R}_k^V}{\Delta_1 + \mathcal{R}_k^V}\right], \tag{21}$$

$$\mathcal{T}_0^{\mathrm{Jac}} = \frac{1}{2}\,\mathrm{Tr}\left[\frac{d_t \mathcal{R}_{S_1}^V}{\Delta_0 + R/3 + \mathcal{R}_k^{S_1}}\right] - \mathrm{Tr}\left[\frac{2 d_t \mathcal{R}_{S_2}^V}{(3\Delta_0 + R)\,\Delta_0 + 4\mathcal{R}_k^{S_2}}\right]. \tag{22}$$

The right-hand side of (18) has been subdivided into contributions coming from fields of different spins. The first two come from the physical spin-2 traceless part of the metric and the spin-0 trace of the metric, as the reader can see by using (14) and (15) in (3). The last two are spin-1 and spin-0 parts coming from field redefinitions.

(18) 的右侧已按不同自旋场的贡献拆分。读者可将 (14)(15) 代入 (3) 验证: 前两项分别来自度规的物理自旋 2 无迹部分和度规的自旋 0 迹部分，最后两项则来自场重定义的自旋 1 和自旋 0 部分。

Cutoff Functions

截断函数

One place where crucial differences occur between the different implementations is in the choice of cutoff $\mathcal{R}_k$. There is quite a lot of freedom as these functions only need to satisfy a few key properties which ensure that they behave like momentum-dependent mass terms suppressing low-momentum modes:

不同实现方案的一个关键差异就在于截断函数 $\mathcal{R}_k$ 的选择。这类函数仅需满足少数保证其能如动量依赖质量项一般压制低动量模式的核心性质，因此选择自由度相当大:

$$\lim_{p^2\to 0} \mathcal{R}_k(p^2) > 0,\ \lim_{p^2\to\infty} \mathcal{R}_k(p^2) = 0,\ \lim_{k\to 0} \mathcal{R}_k(p^2) = 0. \tag{23}$$

The first two conditions ensure that we integrate out the UV modes first and ignore the IR modes. The last condition ensures that we are left with the standard definition of the effective action once the cutoff scale is sent to zero.

前两个条件确保我们先积出紫外模式、忽略红外模式；最后一个条件确保当截断尺度趋近于零时，我们能得到有效作用量的标准定义。

An apparently attractive strategy is to choose cutoffs that simplify the flow equations as much as possible. ”Adaptive cutoffs” are introduced partly with that aim [22-24, 30-41]. They implement the following rule for all appearances of the Laplacian operator $-\nabla^2$:

一个看起来很有吸引力的策略是选择能尽可能简化流方程的截断。引入“自适应截断”部分正是为了这个目标 [22-24, 30-41]。对于拉普拉斯算子 $-\nabla^2$ 的所有出现形式，它们遵循以下规则:

$$-\nabla^2 \mapsto -\nabla^2 + k^2 r\left(-\nabla^2/k^2\right), \tag{24}$$

where $r(z)$ is a cutoff profile function.

其中 $r(z)$ 是截断轮廓函数。

Such a choice also seemingly solves an awkward feature of Euclidean quantum gravity, which is that the Euclidean signature Einstein-Hilbert action (1) has a wrong-sign kinetic term and propagator for $\bar{h}$, the so-called conformal instability [63]. This can be seen in the negative coefficient for Δ_0 in (20) in this case. By implementing (24), the cutoff automatically adapts to this wrong sign, so that it continues to modify the propagator in the intended way: by adding a momentum-dependent mass term. Indeed, if this were not done, the cutoff and kinetic term would have opposite signs, resulting in a singular propagator. However, this trick does not entirely cure the problem since it results in poor asymptotic (large R) behavior. This issue will be briefly touched on below and in section "Flow Equations with Adaptive Cutoff." For further discussion, see Refs. [2,43,54,64-66].

这类选择表面上还解决了欧几里得量子引力的一个棘手问题: 欧几里得号差的爱因斯坦-希尔伯特作用量 (1) 中 $\bar{h}$ 的动能项和传播子符号错误，也就是所谓的共形不稳定性 [63]，这可以从该情况下 (20) 式中 Δ_0 的负系数看出。通过实现 (24) 式，截断会自动适配这个错误符号，因此它能继续以预期方式修改传播子: 添加一个动量依赖质量项。事实上，如果不这么做，截断和动能项符号相反，就会得到奇异传播子。然而，这个技巧并没有完全解决问题，因为它会导致不良的渐近 (大 R) 行为。我们会在下文中和“带自适应截断的流方程”一节简要讨论这个问题。进一步讨论参见文献 [2,43,54,64-66]。

Technically, the above replacement rule is implemented by setting

从技术上讲，上述替换规则通过设置实现:

$$\mathcal{R}_k^{\phi} = \Gamma_k^{(2)}\left[-\nabla^2 + k^2 r\left(-\nabla^2/k^2\right)\right] - \Gamma_k^{(2)}\left[-\nabla^2\right], \tag{25}$$

for each mode ϕ , so that the desired effect is created for $\Gamma_k^{(2)}\left[-\nabla^2\right] + \mathcal{R}_k$ in the flow equation (3). Notice that the cutoff function is then of the same form as the Hessian elements themselves and thus now also depends on $f(R)$. This has a particular consequence for the scalar $\bar{h}$ mode, since $\Gamma_{\bar{h}\bar{h}}^{(2)}$ contains $f_k''(R)$, cf. Eq. (15). It means that plugging this type of cutoff into the flow equation will result in the appearance of $R f_k'''(R)$, due to the presence of $d_t \mathcal{R}_k^{\bar{h}}$ in the numerator in (20) and the definition (4) of $f_k(R)$. This makes the flow equation a third-order differential equation, which unfortunately lacks the powerful properties found in a second-order formulation (as covered in section "Flow Equations with Adaptive Cutoff"). Furthermore, the factor of R leads to a so-called fixed singularity at $R = 0$. Third-order formulations suffer from further fixed singularities and, as already mentioned, poor asymptotic behavior, this latter leading to continuous eigenoperator spectra [54]. These problems will be further covered in section "Flow Equations with Adaptive Cutoff."

对每个模式 ϕ，从而在流方程 (3) 中为 $\Gamma_k^{(2)}\left[-\nabla^2\right]+\mathcal{R}_k$ 产生预期效果。注意此时截断函数与黑塞矩阵元本身形式相同，因此现在也依赖于 $f(R)$。这对标量 $\bar{h}$ 模式有特殊影响，因为 $\Gamma_{\bar{h}\bar{h}}^{(2)}$ 包含 $f_k''(R)$，参见式 (15)。这意味着将这类截断代入流方程后，由于 (20) 式分子中存在 $d_t\mathcal{R}_k^{\bar{h}}$，结合 $f_k(R)$ 的定义 (4)，会得到 $Rf_k'''(R)$。这会让流方程成为三阶微分方程，不幸的是，它不具备二阶形式 (见“带自适应截断的流方程”一节) 的优良性质。此外，因子 R 会在 $R=0$ 处产生所谓的固定奇点。三阶形式还存在更多固定奇点，且如前所述，具有不良的渐近行为，后者会导致连续本征算符谱 [54]。这些问题会在“带自适应截断的流方程”一节进一步讨论。

When using an adaptive cutoff, the cutoff profile function $r\left(-\nabla^2/k^2\right)$ is almost always chosen to be the ”optimized” profile [67]

使用自适应截断时，截断轮廓函数 $r\left(-\nabla^2/k^2\right)$ 几乎总是被选为“优化”轮廓 [67]

$$r(z)=(1-z)\,\theta(1-z)\,. \tag{26}$$

The advantage of using this setup is that $d_t\mathcal{R}_k \propto \theta\left(1+\nabla^2/k^2\right)$ and thus the eigenvalues of $-\nabla^2$ are restricted to be less than k^2. This means that in denominators one can simply ignore the θ and thus $k^2 r\left(-\nabla^2/k^2\right)\equiv k^2+\nabla^2$. Therefore, the net effect in denominators is just to replace $\Gamma_k^{(2)}\left[-\nabla^2\right]$ with $\Gamma_k^{(2)}\left[k^2\right]$, massively simplifying the computation of space-time traces.

这种设置的优势在于，$d_t\mathcal{R}_k \propto \theta\left(1+\nabla^2/k^2\right)$ 因而 $-\nabla^2$ 的本征值都被限制为小于 k^2。这意味着在分母中可以直接忽略 θ，进而忽略 $k^2 r\left(-\nabla^2/k^2\right)\equiv k^2+\nabla^2$。因此，分母中的最终效果仅为用 $\Gamma_k^{(2)}\left[k^2\right]$ 替换 $\Gamma_k^{(2)}\left[-\nabla^2\right]$，极大简化了时空迹的计算。

The second-order formulation [42,43] chooses a non-adaptive cutoff function of the form

二阶表述 [42,43] 选择了如下形式的非自适应截断函数

$$\mathcal{R}_k^{\phi}=k^{m_\phi}c_\phi r\left(\Delta_s+\alpha_s R\right) \tag{27}$$

where s is the spin of the mode ϕ, m_ϕ is set such that the cutoff has the same dimension as $\Gamma^{(2)}$ for this mode, and c_ϕ is a number. In this chapter, the c_ϕ will be taken to be positive for all fields. This is a problem for developing solutions $f_k(R)$ that approximate the perturbative quantization of the Einstein-Hilbert action (1) because the $\bar{h}$ Hessian has the wrong sign there (as noted above). But again the alternative choice $c_{\bar{h}}<0$ leads to poor asymptotic behavior at large R, resulting in a continuous spectrum of eigenoperators [43].

其中 s 是模式的自旋，ϕ, m_ϕ 被设置为使得截断量与该模式下的 $\Gamma^{(2)}$ 量纲相同，且 c_ϕ 是一个数值。本章中，对所有场均取 c_ϕ 为正。这对开发近似爱因斯坦-希尔伯特作用量 (1) 微扰量子化的解 $f_k(R)$ 而言存在问题，因为此处 $\bar{h}$ 黑塞矩阵符号错误 (上文已指出)。但如果选择另一选项 $c_{\bar{h}}<0$，又会在大 R 处出现糟糕的渐近行为，进而产生本征算符的连续谱 [43]。

Notice that the cutoffs (27) have been chosen to depend on Δ_s, rather than simply the $-\nabla^2$ part [33], and furthermore include an ”endomorphism,” a curvature correction with endomorphism coefficient α_s [42]. In Refs. [42,43], the traces are computed directly as a sum over modes. The α_s are there to ensure that

注意，式 (27) 的截断量被选为依赖于 Δ_s，而非仅依赖 $-\nabla^2$ 部分 [33]，此外还包含一个“自同态”，即带有自同态系数 α_s 的曲率修正 [42]。在文献 [42,43] 中，迹直接通过模式求和计算。引入 α_s 是为了确保

$$\Delta_s + \alpha_s R > 0, \tag{28}$$

for all modes, which in turn ensures that they are all integrated out as $k \to 0$ and that the flow equation does not suffer from fixed singularities. For these non-adaptive cutoffs, the optimized cutoff profile (26) brings no particular advantage. In fact, on a sphere, the trace is a discrete sum, and sharp cutoff profiles would lead to a staircase behavior [33], with an ill-defined limit as $R \to 0$. Hence, a smooth (infinitely differentiable) cutoff profile is used, such as [10]

对所有模式成立，这反过来保证当 $k \to 0$ 时所有模式都被积出，且流方程不存在固定奇点。对于这类非自适应截断，优化截断轮廓 (26) 并没有特殊优势。事实上，在球面上迹是离散求和，陡锐截断轮廓会导致阶梯行为 [33]，且当 $R \to 0$ 时极限没有良好定义。因此一般采用光滑 (无穷可微) 的截断轮廓，例如文献 [10] 中的

$$r(z) = \frac{z}{\exp(az^b) - 1}, \quad a > 0, b \geq 1. \tag{29}$$

Flow Equations with Adaptive Cutoff

带自适应截断的流方程

In those formulations that use an adaptive cutoff, space-time traces are evaluated using a heat-kernel asymptotic expansion, apart from Ref. [33] which uses a direct spectral sum together with a smoothing procedure (to get over the aforementioned staircase problem). As an illustration, the result of the earliest four such formulations [30-32] for the flow of $f \equiv f(R,t)$ on a four-sphere can be summarized as

在那些采用自适应截断的表述中，时空迹除了文献 [33](该文献采用直接谱和结合平滑处理来解决前文提到的阶梯问题) 外，均使用热核渐近展开计算。作为示例，最早的四个此类表述 [30-32] 得到的四维球面上 $f \equiv f(R,t)$ 流的结果可总结为

$$384\pi^2 (\partial_t f + 4f - 2Rf') =$$

$$\left[5R^2\theta\left(1-\frac{R}{3}\right) - \left(12 + 4R - \frac{61}{90}R^2\right)\right]\left[1-\frac{R}{3}\right]^{-1} + \sum$$

$$+\left[10R^2\theta\left(1-\frac{R}{4}\right) - R^2\theta\left(1+\frac{R}{4}\right) - \left(36 + 6R - \frac{67}{60}R^2\right)\right]\left[1-\frac{R}{4}\right]^{-1}$$

$$+\left[(\partial_t f' + 2f' - 2Rf'')\left(10 - 5R - \frac{271}{36}R^2 + \frac{7249}{4536}R^3\right)\upsilon\right.$$

$$\left.+f'\left(60 - 20R - \frac{271}{18}R^2\right)\right]\left[f + f'\left(1-\frac{R}{3}\right)\right]^{-1}$$

$$+\frac{5R^2}{2}\left[(\partial_t f' + 2f' - 2Rf'')\left\{r\left(-\frac{R}{3}\right) + 2r\left(-\frac{R}{6}\right)\right\}\right.$$

$$\left.+2f'\theta\left(1+\frac{R}{3}\right) + 4f'\theta\left(1+\frac{R}{6}\right)\right]\left[f + f'\left(1-\frac{R}{3}\right)\right]^{-1}$$

$$+\left[(\partial_t f' + 2f' - 2Rf'')\, f'\left(6 + 3R + \frac{29}{60}R^2 + \frac{37}{1512}R^3\right)\right.$$

$$+(\partial_t f'' - 2Rf''')\left(27 - \frac{91}{20}R^2 - \frac{29}{30}R^3 - \frac{181}{3360}R^4\right)$$

$$\left.+f''\left(216 - \frac{91}{5}R^2 - \frac{29}{15}R^3\right) + f'\left(36 + 12R + \frac{29}{30}R^2\right)\right]$$

$$\times\left[2f + 3f'\left(1 - \frac{2}{3}R\right) + 9f''\left(1-\frac{R}{3}\right)^2\right]^{-1}. \tag{30}$$

Here, the function r is the optimized cutoff profile (26), which also leads to the appearance of the step functions (a.k.a. Heaviside θ functions). In Ref. [32], the equation is adapted to polynomial truncations only, which means that the step functions are all set to one. The first two lines of the right-hand side are independent of $f(R,t)$ and encapsulate the contributions from the ghosts, auxiliaries, ξ_μ , and σ . Here, we have introduced the term $\sum$. The third and fourth lines arise from $h^T_{\mu\nu}$, while the final ratio is the contribution from h . Unphysical modes are isolated differently in these implementations, but the changes can be summarized in the different expressions

此处，函数 r 是优化截断轮廓 (26)，它也会导致阶跃函数 (即海维赛德 θ 函数) 出现。文献 [32] 中，该方程仅适配多项式截断，这意味着所有阶跃函数都被设为 1。等式右侧前两行与 $f(R,t)$ 无关，包含了鬼场、辅助场、ξ_μ 和 σ 的贡献。此处我们引入了项 $\sum$ 。第三行和第四行来自 $h^T_{\mu\nu}$ ，最后一项比值是 h 的贡献。在这些实现中，非物理模式的分离方式不同，但差异可归纳为不同的表达式

$$\sum = 0,\ 10R^2\theta\left(1-\frac{R}{3}\right),\ -\frac{10R^2\left(R^2 - 20R + 54\right)}{(R-3)(R-4)},\ \frac{10\,(11R-36)}{(R-3)(R-4)}.$$

(31)

The first, third, and fourth options are derived in Refs. [30, 32], while the second option comes from Ref. [31]. We have suppressed some other details; for more discussion, see Ref. [54].

第一、第三和第四种选项由文献 [30, 32] 推导得出，第二种选项来自文献 [31]。我们省略了其他一些细节；更多讨论参见文献 [54]。

Setting $\partial_t f = 0$ in the above turns this flow equation into the differential equation that must be satisfied by a fixed point $f(R)$. It is a highly non-linear third-order ODE (ordinary differential equation). In the formulation [31], the appearance of the θ functions, explicitly and in r , will result in jumps in $f'''(R)$ across the point where they switch on or off, but this can be accommodated.

将上述表达式中的 $\partial_t f = 0$ 设定后，该流方程就转变为不动点 $f(R)$ 必须满足的微分方程。这是一个高度非线性的三阶常微分方程 (ODE)。在表述 [31] 中，θ 函数显式出现在 r 中，会导致 $f'''(R)$ 在函数开闭点处发生跳变，但这一问题可以处理。

A more important and generic feature is the existence of fixed and moveable singularities. These concepts come from the mathematics of analysis of ODEs. To discuss them, it is helpful to cast the fixed-point ODE in ”normal” form:

一个更重要的通用特征是固定奇点和可动奇点的存在。这些概念源自常微分方程分析的数学理论。为讨论它们，将不动点常微分方程整理为「标准」形式会更方便:

$$f'''(R) = rhs, \tag{32}$$

where rhs (right-hand side) contains no f''' terms. A Taylor expansion about some generic point R_p takes the form

其中 (等式右侧的) rhs 不含 f''' 项。对任意普通点 R_p 做泰勒展开可得

$$f(R) = f(R_p) + (R - R_p) f'(R_p) + \frac{1}{2}(R - R_p)^2 f''(R_p) + \frac{1}{6}(R - R_p)^3 f'''(R_p) + \cdots.$$

(33)

Since (32) determines the fourth coefficient in terms of the first three, we see that typically (33) provides a series solution depending on three continuous real parameters, here

由于 (32) 可由前三个系数确定第四个系数，因此我们发现，(33) 给出的级数解通常依赖于三个连续实参数，即

$$f(R_p),\ f'(R_p) \ \text{ and } \ f''(R_p), \tag{34}$$

with some finite radius of convergence ρ whose value also depends on these parameters. Therefore, the standard mathematical result is recovered that around a generic point R_p , there is some domain $\mathcal{D} = (R_p - \rho, R_p + \rho)$ in which there is a three-parameter set of well-defined solutions. From here, one can try to extend the solution to a larger domain, e.g., by matching to a Taylor expansion about another point within $\mathcal{D}$. A typical problem, seen also in the LPA and the derivative expansion [48-50,52,53] and in the second-order formulation [42,43], is that eventually, at some point $R = R_c$, dependent on the parameters, the denominator of rhs develops a zero, so that as $R \to R_c$,(32) implies

其收敛半径 ρ 有限，且收敛半径的数值也依赖于这三个参数。因此我们可以得到标准数学结论: 在任意普通点 R_p 附近，存在某个区域 $\mathcal{D} = (R_p - \rho, R_p + \rho)$，区域内存在一个含三参数的良定解族。由此，我们可以尝试将解延拓到更大的区域，例如通过与 $\mathcal{D}$ 内另一个点附近的泰勒展开匹配来实现。在 LPA、导数展开 [48-50,52,53] 以及二阶表述 [42,43] 中都能看到一个典型问题: 最终在依赖于参数的某点 $R = R_c$ 处，rhs 的分母会变为零，因此当 $R \to R_c$ 时，(32) 表明

$$f'''(R) = 2c/(R - R_c) + \cdots, \tag{35}$$

where c is some constant and the ellipsis contains the non-singular part. Integrating this, we see that the solution typically ends in a moveable singularity, of form

其中 c 是某个常数，省略号包含非奇异部分。对其积分后我们发现，解通常终止于一个如下形式的可动奇点:

$$f(R) \sim c(R - R_c)^2 \ln |R - R_c| , \tag{36}$$

where ”~” means that less singular parts are neglected.

其中「~」表示忽略低奇异阶部分。

As already mentioned, fixed-point equations derived with adaptive cutoff present another challenge in that they also have fixed singular points R_c . These correspond in *rhs* to explicit algebraic poles in R , where the domain of interest is $R \geq 0$ since the equations apply to the four-sphere. Whatever the formulation, there is always one fixed singularity $R_c = 0$, which is unavoidable when using an adaptive cutoff as we have seen [33, 54] . Different formulations have different numbers and positions for the other fixed singularities (see, e.g., the discussion in Refs. [38,57]), but there is always at least one more. Inspecting the example (30), we see that f''' appears once in the penultimate line in Eq. (30), where it is multiplied by the polynomial

如前所述，由自适应 cutoff 导出的不动点方程存在另一难点: 它们也存在固定奇点 R_c 。这在 *rhs* 中对应 R 中的显式代数极点，由于方程适用于四维球面，因此我们关注的定义域是 $R \geq 0$ 。无论采用何种形式，始终存在一个固定奇点 $R_c = 0$ ， 正如我们所见 [33, 54] ， 使用自适应 cutoff 时这无法避免。不同形式下其余固定奇点的数量和位置各不相同 (例如参见文献 [38,57] 中的讨论)， 但始终至少额外存在一个。观察示例 (30) 我们可以看到， f''' 出现在式 (30) 倒数第二行， 且乘有多项式

$$R\left(27 - \frac{91}{20}R^2 - \frac{29}{30}R^3 - \frac{181}{3360}R^4\right). \tag{37}$$

Thus, rearranging the fixed-point equation into normal form (32) results in poles from the zeroes of this polynomial. Two of these are in the required domain, namely, at $R_c = 0$ and $R_c = 2.0065$. There are also two further single poles, at $R_c = 3$ and $R_c = 4$, from the first two lines of the right-hand side of (30).

因此，将不动点方程整理为范式 (32) 后，该多项式的零点会产生极点。其中两个极点位于所需定义域内，分别在 $R_c = 0$ 和 $R_c = 2.0065$ 处。此外，式 (30) 右侧前两行还另外产生两个单极点，分别位于 $R_c = 3$ 和 $R_c = 4$ 处。

As R approaches one of these R_c, f will end at a singularity of form (36) unless the f -dependent parts in *rhs* are tuned so as to conspire to cancel the pole. Substituting the Taylor expansion (33), with $R_p = R_c$, one sees that this requirement forces some generally non-linear combination of $f(R_c)$, $f'(R_c)$, and $f''(R_c)$ to vanish. Thus, a fixed singularity imposes a constraint on the solution, reducing the number of free parameters by one.

当 R 接近这些点之一时，除非调整 *rhs* 中依赖 f 的部分，使其共同抵消极点，否则 R_c, f 会终止于形式为 (36) 的奇点。代入带 $R_p = R_c$ 的泰勒展开 (33) 可以发现， 该条件要求 $f(R_c), f'(R_c)$ 和 $f''(R_c)$ 的某个一般非线性组合为零。因此，固定奇点会对解施加约束，使自由参数的数量减少一个。

The inevitable fixed singularity at $R_c = 0$ can thus be seen as restoring consistency since it reduces the three-parameter set of solutions to a two-parameter set, in agreement in this respect with what is obtained from the non-adaptive cutoff second-order formulation.

因此，位于 $R_c = 0$ 处不可避免的固定奇点可被视为恢复了自洽性: 它将三参数解集约化为两参数解集，这一点与非自适应 cutoff 二阶形式得到的结果一致。

Unfortunately, since there are further three fixed singularities, these equations are over-constrained, and thus, there are no fixed-point solutions $f(R)$ that are valid over the whole range $R \geq 0$.

遗憾的是，由于还额外存在三个固定奇点，这些方程是过约束的，因此不存在在整个 $R \geq 0$ 范围内都有效的不动点解 $f(R)$ 。

However, these fixed singularities are artifacts of the regularization procedure: it is possible to move them and eliminate most of them. Benedetti and Caravelli were the first to realize this, and we will refer to their version [33] as the "BC" formulation. Before regularization, the Jacobian trace (21) has a denominator that vanishes if Δ_1 vanishes. Likewise, the Jacobian trace (22) has a denominator that vanishes when Δ_0 vanishes. Recalling the form (17) of the Δ_s and that the net effect of the adaptive optimized cutoff is to replace $-\nabla^2$ with k^2 in the denominator, we see that these contributions give poles $1/(1-R/4)$ and $1/(1-R/3)$ (after using (4) to scale to dimensionless quantities). These are the poles that are visible in the first two lines of the right-hand side of (30).

但这些固定奇点是正则化过程带来的人工产物: 我们可以移动这些奇点，并消除其中大部分。Benedetti 和 Caravelli 最先意识到这一点，我们将他们文献 [33] 中的版本称为“BC”形式。正则化之前，雅可比迹 (21) 的分母会在 Δ_1 为零时变为零。同理，雅可比迹 (22) 的分母会在 Δ_0 为零时变为零。回顾 Δ_s 的形式 (17)，再结合自适应优化 cutoff 的净效应是将分母中的 $-\nabla^2$ 替换为 k^2 ，我们可以看到，这些贡献会产生极点 $1/(1-R/4)$ 和 $1/(1-R/3)$ (利用 (4) 标度为无量纲量后)。这些正是 (30) 右侧前两行中可见的极点。

BC eliminate them by using an endomorphism, namely, by using $r(\Delta_s)$ instead of $r\left(-\nabla^2\right)$ [33] (a so-called cutoff of type II [32]). Then one is left with the $R_c = 0$ singularity and a fixed singularity at some positive R_c which is due to the fact that the $\bar{h}$ trace vanishes there [33, 54] . These fixed singularities thus reduce $f(R)$ solutions to a one-parameter set.

BC 通过使用自同构消除了这些极点，具体而言就是用 $r(\Delta_s)$ 代替 $r\left(-\nabla^2\right)$ [33](即所谓的 II 型 cutoff[32])。之后仅剩下 $R_c = 0$ 奇点和某个正 R_c 处的固定奇点，该奇点产生的原因是 $\bar{h}$ 迹在此处变为零 [33, 54] 。因此这些固定奇点将 $f(R)$ 解约化为单参数集。

Now there is still the danger of encountering a moveable singularity (36), and this imposes further restrictions on the remaining parameter. Such a singularity can appear at any value of R and in particular at large R where the equations can then be solved analytically by developing the solution as an asymptotic expansion. In scalar field theory [48-53] and in the second-order formulation [43], what is found is that this asymptotic expansion has less than the full number of parameters expected. One can also show that the missing parameters are associated with fast growing perturbations that are incompatible with an asymptotic solution. In this way, it is possible to deduce analytically the number of constraints that moveable singularities are responsible for imposing.

目前仍存在遇到可动奇点 (36) 的风险，这会对剩余参数施加进一步限制。这类奇点可出现在 R 的任意取值处，尤其会出现在大 R 处，在此处可通过将解展开为渐近展开来解析求解方程。在标量场论 [48-53] 和二阶表述 [43] 中，研究发现该渐近展开所含参数数量少于预期的完整参数数量。还可证明，缺失的参数与不符合渐近解的快速增长微扰相关。通过这种方式，可以解析推导出可动奇点施加的约束数量。

The result for scalar field theory is that the parameters are fixed, typically to a handful of values [48, 49, 53] , corresponding to a finite set of fixed points, or in special cases a discrete infinity of fixed points [50]. However, there is at this stage also the possibility that there are no fixed-point solutions. The actual number of solutions then needs to be determined numerically (Although some may be found analytically, e.g., the Gaussian fixed point or special cases [40]). We will see this at work in the second-order formulation in section ”Fixed-Point Solutions” where we describe in detail how to find asymptotic solutions $f_{\text{asy}}(R)$.

标量场论的结果是，参数通常被固定为少量离散值 [48, 49, 53] ，对应有限个不动点，在特殊情况下对应无穷多个离散不动点 [50]。但在此阶段也可能不存在不动点解。因此解的实际数量需要通过数值确定 (部分解可通过解析得到，例如高斯不动点或特殊情况 [40])。我们会在“不动点解”一节的二阶表述中看到这一过程，其中我们会详细描述如何寻找渐近解 $f_{\text{asy}}(R)$ 。

Unfortunately, for third-order formulations, asymptotic analysis typically does not find sufficient constraints [57]. For example, for the BC formulation, the asymptotic solution turns out to have the maximum three parameters [54]:

遗憾的是，对于三阶表述，渐近分析通常无法得到足够的约束 [57]。例如，对于 BC 表述，渐近解最多只有三个自由参数 [54]:

$$f_{\text{asy}}(R) = AR^2 + R\left\{\frac{3}{2}A + B\cos\ln R^2 + C\sin\ln R^2\right\} + \cdots, \tag{38}$$

where the ellipses stand for asymptotic corrections with lower powers of R and the three parameters are restricted only by the inequality

其中省略号代表 R 低次幂的渐近修正，三个参数仅受不等式约束

$$\frac{121}{20}A^2 > B^2 + C^2 \tag{39}$$

Thus, one still expects to find one-parameter sets (i.e., lines) of global solutions $f(R)$ in this case, and that is exactly what is found by careful numerical analysis [54]. Asymptotic analysis also shows that the BC formulation has continuous eigenoperator spectra. Initially, it was suggested that these effects can be attributed to the fact that all eigenoperators are redundant if the equation of motion (16) for the fixed point $f(R)$ has no solution for R in the required range $R \geq 0$ [55]. But it is now clear that the poor behavior is again associated with the scalar mode $\bar{h}$ [38, 54, 64] and is one more malign effect of the conformal instability [54, 63, 64] . In fact, precisely these problems reappear in the second-order formulation if one chooses $c_{\bar{h}} < 0$, as already mentioned in section ”Cutoff Functions.”

因此，在这种情况下仍可预期存在单参数族 (即线) 整体解 $f(R)$ ，而精细数值分析确实得到了这样的结果 [54]。渐近分析还表明 BC 表述存在连续本征算子谱。最初有观点认为，这类效应可归因于: 若不动点的运动方程 (16) $f(R)$ 在要求的范围 $R \geq 0$ 内不存在 R 的解，则所有本征算子都是冗余的 [55]。但目前已经明确，这种不良性质仍与标量模式 $\bar{h}$ [38, 54, 64] 相关，是共形不稳定性 [54, 63, 64] 的又一有害效应。事实上，正如“截断函数”一节已经提到的，若选择 $c_{\bar{h}} < 0$ ，这些问题会在二阶表述中再次出现。

As emphasized in Ref. [57], asymptotic analysis plays three powerful roles. Firstly, as just sketched and discussed in detail in section ”Asymptotic Analysis,” it allows one to deduce the dimension of the solution space. Secondly, the asymptotic solution provides a way to validate numerical solutions since if one can integrate out far enough, the numerical solution should match the asymptotic solution, allowing a reliable determination of the asymptotic parameters.

正如文献 [57] 所强调的，渐近分析发挥三个重要作用。第一，如刚才概述并在“渐近分析”一节详细讨论的，它可以推导出解空间的维数。第二，渐近解为数值解提供了验证方法: 如果积分进行得足够远，数值解应当与渐近解匹配，从而可以可靠地确定渐近参数。

Finally, the asymptotic solution actually contains only the physical part of the fixed-point effective action. To see this, we need to return temporarily to labelling scaled quantities with a tilde and recall that the effective infrared cutoff k is added by hand such that the physical Legendre effective action is recovered only in the limit that this cutoff $k \to 0$. This must be done while holding the physical quantities such as R fixed, rather than scaled quantities $\widetilde{R}$. In normal field theory, e.g., scalar field theory, the analogous object is the universal scaling equation of state, which for a constant field precisely at the fixed point takes the simple form

最后，渐近解实际上仅包含不动点有效作用量的物理部分。要理解这一点，我们需要暂时回到用波浪号标记标度量的约定，并回顾: 有效红外截断 k 是手动添加的，只有当该截断 $k \to 0$ 的极限下才能恢复物理勒让德有效作用量。这一步必须在固定 R 这类物理量而非标度量 $\widetilde{R}$ 的前提下进行。在普通场论 (例如标量场论) 中，对应的对象是普适标度物态方程，对于恰好位于不动点的常数场，它具有简单形式:

$$V(\varphi) = A\varphi^{d/d_\varphi}, \tag{40}$$

where d is the space-time dimension and d_φ is the full scaling dimension of the field (i.e., incorporating also the anomalous dimension). In the current case, we keep fixed the constant background scalar curvature R . Thus, by (2) and (4), the only physical part of the fixed-point action in this approximation is

其中 d 是时空维数，d_φ 是场的完整标度维数 (即也包含了反常维度)。在本文的情形中，我们保持常数背景标量曲率 R 固定。因此，根据 (2) 和 (4)，该近似下不动点作用量唯一的物理部分是:

$$f(R)\big|_{\text{phys}} = \lim_{k\to 0} k^4 \widetilde{f}\left(R/k^2\right) = \lim_{k\to 0} k^4 \widetilde{f}_{\text{asy}}\left(R/k^2\right). \tag{41}$$

For example, from (38), for the BC formulation, one finds

例如，从 (38) 出发，可以得到 BC 表述的结果:

$$f(R)|_{\text{phys}} = AR^2. \tag{42}$$

This is invariant under changes of scale as it must be and is a sensible answer for the scaling equation of state precisely at the fixed point. We will find the same answer from the second-order formulation.

它满足标度变换不变性的要求，恰好给出了不动点处标度物态方程的合理解。我们会从二阶表述得到相同的结果。

We still have the problem that since there are one-parameter sets of fixed-point solutions, A is not fixed. In third-order formulations, one can use the ability to add endomorphisms to try to patch this up [38], but asymptotic analysis then shows there is actually a whole zoo of possibilities for the scaling equation of state and dimension of the solution space, depending on parameter choices in the endomorphisms [57]. One can also try to extend the solution to negative R . This does reduce the solution space of the BC formulation to a discrete set, but that set appears to be empty since no numerical solutions were then found [54]. A more careful version of this strategy is also used in the second-order formulation.

我们仍然面临这样的问题: 由于不动点解存在单参数集合，A 并未固定。在三阶表述中，我们可以利用添加自同态的方式尝试修补这一问题 [38]，但渐近分析表明，依赖于自同态中的参数选择，标度物态方程与解空间维度实际上存在多种多样的可能性 [57]。我们也可以尝试将解延拓到负的 R 。这确实能将 BC 表述的解空间约化为离散集合，但该集合似乎是空集，因为目前没有找到任何数值解 [54]。这种策略的更精细版本也被用于二阶表述中。

Actually, one can question whether the large $\widetilde{R} = R/k^2$ regime makes physical sense [37, 38, 40] . The problem arises when the cutoff depends on modified Laplacians, e.g., as in (28), where the endomorphism is added to ensure that the minimum eigenvalue is positive. It is most easily seen if we take a sharp (step function) cutoff profile and write the minimum eigenvalue as $R\lambda_{\min}$. Then once $k^2 < R\lambda_{\min}$, i.e., $\widetilde{R} > 1/\lambda_{\min}$, there are no more modes to be integrated out. This means that the functional behavior in this large $\widetilde{R}$ regime is meaningless since it is not describing any actual changes. However, the physical Legendre effective action is only reached by taking $k \to 0$, and this argument would appear to imply that such a limit is inherently ill-defined.

实际上，我们可以质疑大 $\widetilde{R} = R/k^2$ 区域是否具有物理意义 [37, 38, 40] 。当截断依赖于修正拉普拉斯算子时就会出现这个问题，例如 (28) 式的情况，其中添加自同态是为了保证最小本征值为正。如果我们取一个锐截断 (阶跃函数) 轮廓，将最小本征值写为 $R\lambda_{\min}$ ，就可以很清楚地看到这一点。当满足 $k^2 < R\lambda_{\min}$ ，即 $\widetilde{R} > 1/\lambda_{\min}$ 时，就不再有需要积出的模式了。这意味着大 $\widetilde{R}$ 区域的泛函行为没有意义，因为它无法描述任何实际的变化。然而，物理勒让德有效作用量只有在取 $k \to 0$ 时才能得到，这一论证似乎暗示该极限本身就是定义不良的。

In fact, this conundrum is another artifact of the single-metric approximation [22]. In reality, one should be integrating out over an ensemble of manifolds described by the fluctuating full metric $\hat{g}_{\mu\nu}$. The Wilsonian RG only makes sense when applied to such an ensemble. Then no matter how small k is, there are always manifolds with sufficiently small curvature that their eigenvalues remain to be integrated out. It is possible to repair the single-metric approximation sufficiently in this case by retaining the scale degree of freedom $h_{\mu\nu} \propto g_{\mu\nu}$ in the fluctuation field dependence and thus regaining an ensemble of manifolds. However, the net result of such a repair is the same type of functional RG equations again, but now with a clear explanation for why the large $\widetilde{R}$ regime should be trusted [22-24].

实际上，这个难题是单度量近似的另一个人为产物 [22]。现实中，我们应当对由涨落全度量 $\hat{g}_{\mu\nu}$ 描述的流形整体进行积分。威尔逊重整化群只有应用于这样的整体才有意义。此时无论 k 多小，总有曲率足够小的流形，其本征值仍有待积出。在这种情况下，我们可以通过在涨落场依赖中保留标度自由度 $h_{\mu\nu} \propto g_{\mu\nu}$，充分修复单度量近似，重新得到流形整体。然而，这种修复的最终结果仍是同一类泛函重整化群方程，不过它现在可以清晰解释为什么大 $\widetilde{R}$ 区域是可信的 [22-24]。

We now abandon third-order formulations and concentrate on a second-order formulation [42,43], which in almost all respects has more promising behavior.

我们现在放弃三阶表述，专注于二阶表述 [42,43]，几乎从各方面来看，二阶表述都具有更理想的性质。

Evaluating Traces

计算迹

In the formulation [42,43], the traces are evaluated by a direct spectral sum. In common with the rest of the literature, one chooses a (globally) maximally symmetric background manifold. There are three to choose from: the four-sphere $\mathbb{S}^4$, which has a finite volume and positive curvature, so the spectrum of the allowed modes forms a discrete set that have to be summed over; the hyperboloid $\mathbb{H}^4$ which has negative curvature and infinite volume so the spectrum is continuous; and finally the flat space $\mathbb{R}^4$, which is a limiting case for both of the two previous manifolds when $R \to 0$. As we will see, they all need to be considered. Actually, they become smoothly joined together in an ensemble which thus allows the same flow equation to be defined over the entire domain $-\infty < R < \infty$.

在文献 [42,43] 的公式化表述中，迹通过直接谱求和计算。与现有文献的通用做法一致，我们选择 (全局) 极大对称背景流形。共有三类流形可供选择: 四维球 $\mathbb{S}^4$，它体积有限且曲率为正，因此允许模式的谱构成离散集合，需要对其求和；双曲面 $\mathbb{H}^4$，它曲率为负且体积无限，因此谱是连续的；最后是平直空间 $\mathbb{R}^4$，它是前两类流形在 $R \to 0$ 情况下的极限。我们将会看到，这三类都需要被考虑。实际上，它们在系综中平滑连接在一起，因此可以在整个区域 $-\infty < R < \infty$ 上定义同一个流方程。

Sphere

球体

On the sphere, the traces are evaluated using

在球体上，迹通过以下方法计算

$$\operatorname{Tr} W(\Delta_s) = \sum_n D_{n,s} W(\lambda_{n,s}) \tag{43}$$

where $\lambda_{n,s}$ are eigenvalues of the Δ_s defined in (17) and $D_{n,s}$ are their multiplicities. Explicit values are shown in Table 1 [33]. There are a few caveats. Not all the modes contribute in the sum, for example, vectors satisfying $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ and the scalar modes $\sigma =$ constant. Because of this, the tensor mode and the vector mode sums start at $n = 2$, the scalar mode of the Jacobian starts at $n = 1$, and the $\bar{h}$ mode starts at $n = 0$. Now the requirement (28) means that $\lambda_{n,s} + \alpha_s R > 0$ must be satisfied. For the tensor and vector modes, it is sufficient to set $\alpha_2 = \alpha_1 = 0$; however, from Table 1, we see that we must have $\alpha_0 > 1/3$.

其中 $\lambda_{n,s}$ 是式 (17) 中定义的 Δ_s 的特征值，$D_{n,s}$ 是其重数。具体数值见表 1[33]。此处有几点注意事项: 并非所有模式都对求和有贡献，例如满足 $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ 的向量和为常数的标量模式 $\sigma =$ 。因此，张量模式和向量模式的求和从 $n = 2$ 开始，雅可比矩阵的标量模式从 $n = 1$ 开始，$\bar{h}$ 模式从 $n = 0$ 开始。此时条件 (28) 要求必须满足 $\lambda_{n,s} + \alpha_s R > 0$ 。对于张量模式和向量模式，只需设 $\alpha_2 = \alpha_1 = 0$ ；但从表 1 可知，我们必须满足 $\alpha_0 > 1/3$ 。

Table 1 Values of the multiplicities and eigenvalues for evaluating the traces

表 1 计算迹所用的重数与特征值

Spin s	Eigenvalue $\lambda_{s,n}$	Multiplicity $D_{n,s}$
0	$\frac{n(n+3)-4}{12}R$	$\frac{(n+2)(n+1)(2n+3)}{6}$
1	$\frac{n(n+3)-4}{12}R$	$\frac{n(n+3)(2n+3)}{2}$
2	$\frac{n(n+3)}{12}R$	$\frac{5(n+4)(n-1)(2n+3)}{6}$

Hyperboloid

双曲面

As already mentioned, the hyperboloid has a negative curvature, an infinite volume, and a continuous spectrum of eigenvalues. The traces on this manifold are evaluated using [68]

如前所述，双曲面具有负曲率、无限体积，且特征值为连续谱。该流形上的迹可通过文献 [68] 计算

$$\mathrm{Tr}\, W(\Delta_s) = \frac{2s+1}{8\pi^2} \int d^4x\sqrt{g}\left(-\frac{R^2}{12}\right)^2 \int_0^\infty d\lambda \left(\lambda^2 + \left(s+\frac{1}{2}\right)^2\right) \tag{44}$$

$$\times \lambda \tanh(\pi\lambda)\, W\left(\Delta_{\lambda,s}\right) .$$

Even though there is now an infinite volume factor in the flow equation (18), this precise factor also appears above, so the equations still make sense once we cancel this factor from both sides. The eigenvalues of the spectrum are

尽管流方程 (18) 中现在出现了一个无限体积因子，但该因子也已在上文出现，因此我们将该因子从方程两边消去后，方程仍然成立。谱的特征值为

$$\Delta_{\lambda,s} = -\frac{R}{12}\lambda^2 - \beta_s R, \text{ where } \beta_0 = \frac{25}{48}, \beta_1 = \frac{25}{48}, \beta_2 = \frac{9}{48}. \tag{45}$$

Using the same flow equation, and thus the same endomorphism parameters α_s , the requirement (28) must again be satisfied. We can still take $\alpha_2 = \alpha_1 = 0$, but now α_0 also has an upper bound $\alpha_0 < 25/48$.

使用相同的流方程，从而使用相同的自同态参数 α_s ， 条件 (28) 必须再次得到满足。我们仍然可以取 $\alpha_2 = \alpha_1 = 0$ ， 但此时 α_0 也存在一个上界 $\alpha_0 < 25/48$ 。

Flat Space

平坦空间

Finally, evaluating traces on flat space can be achieved by taking the limit as $R \to 0$ from positive or negative side. If we start from the positive side, we first make a substitution $p = n\sqrt{R/12}$ and then take $R \to 0$ while keeping p fixed. All Laplacians then become $\Delta_{n,s} \to p^2$, and p^2 can be identified as the flat space momentum. Plugging in our choice of the cutoff (27), and performing these substitutions, yields

最后，平坦空间的迹计算可以通过从正方向或负方向取 $R \to 0$ 的极限实现。如果我们从正方向出发，首先做替换 $p = n\sqrt{R/12}$ ，随后保持 p 固定取 $R \to 0$ 极限。此时所有拉普拉斯算子都变为 $\Delta_{n,s} \to p^2$ ， 且 p^2 可对应为平坦空间动量。代入我们选取的截断 (27) 并完成这些替换后， 可得

$$\begin{aligned}\partial_t f_k(0) + 4 f_k(0) = \frac{1}{8\pi^2}\int_0^\infty dp\, p^3 \Bigg[& 16 c_{\bar h} \frac{2r(p^2) - p^2 r'(p^2)}{9 f_k''(0)\, p^4 + 3 f_k'(0)\, p^2 + 2 f_k(0) + 16 c_{\bar h} r(p^2)} \\ & + 10 c_T \frac{r(p^2) - p^2 r'(p^2)}{-f_k'(0)\, p^2 - f_k(0) + 2 c_T r(p^2)} - 3 c_V \frac{r(p^2) - p^2 r'(p^2)}{p^2 + c_V r(p^2)} \\ & - 4 c_{S_2} \frac{2r(p^2) - p^2 r'(p^2)}{3p^4 + 4 c_{S_2} r(p^2)} + c_{S_1} \frac{r(p^2) - p^2 r'(p^2)}{p^2 + c_{S_1} r(p^2)} \end{aligned} \tag{46}$$

This same equation is arrived at if we take $R \to 0$ from the negative side by first setting $p = \lambda\sqrt{-R/12}$ on the hyperboloid and holding p fixed. The form of these equations already gives some information about the possible solutions and can help guide numerical searches [43]. In particular, by inspection, it is clear that there are no fixed singularities and choices for $f_k(0), f_k'(0)$ and $f_k''(0)$ can be made that give well-defined non-singular integrals.

如果我们从负方向取 $R \to 0$ 极限，先在双曲面上设定 $p = \lambda\sqrt{-R/12}$ 并保持 p 固定，也能得到相同的方程。这些方程的形式已经给出了关于可能解的部分信息，可为数值搜索提供指导 [43]。特别地，通过观察不难发现，该过程不存在固有奇点，我们可以对 $f_k(0), f_k'(0)$ 和 $f_k''(0)$ 做出合适选择，得到定义明确的非奇异积分。

Fixed-Point Solutions

不动点解

The fixed-point solution to the flow equation $f_k(R) = f(R)$ occurs when $\partial_t f_k(R) = 0$. An advantage of the non-adaptive cutoff is that $\partial_t f_k(R)$ only appears once on the left-hand side of (18), so the fixed-point equation is

流方程 $f_k(R) = f(R)$ 的不动点解出现在 $\partial_t f_k(R) = 0$ 时。非自适应截断的一个优点是，$\partial_t f_k(R)$ 仅出现在式 (18) 的左侧一次，因此不动点方程为

$$2VE(R) = \mathcal{T}_2 + \mathcal{T}_0^{\hbar} + \mathcal{T}_1^{\text{Jac}} + \mathcal{T}_0^{\text{Jac}}. \tag{47}$$

Another crucial advantage is, like (46), inspection of the trace equations (19)-(22) makes clear that there are no fixed singularities any more. The flow equation is nonlinear and very hard to work with, so solving the equations exactly is unfeasible. The strategy is to solve analytically for $f(R)$ around $R = 0$ as a Taylor expansion and around $R = \pm\infty$ by an asymptotic expansion. Then numerical methods can be used to try to patch in a solution that goes smoothly from the Taylor expansion at $R = 0$ to the asymptotic solutions at $R = \pm\infty$.

另一个关键优点是，和式 (46) 一样，通过观察迹方程 (19)-(22) 可知现在不再存在固定奇点。该流方程是非线性的，非常难处理，因此精确求解这些方程是不可行的。我们的策略是: 在 $R = 0$ 附近对 $f(R)$ 做泰勒展开解析求解，在 $R = \pm\infty$ 附近做渐近展开解析求解；随后可使用数值方法尝试拼接出一个解，使其从 $R = 0$ 处的泰勒展开平滑过渡到 $R = \pm\infty$ 处的渐近解。

Asymptotic Analysis

渐近分析

We now explain in detail how to develop asymptotic solutions, using these equations as an example. In these large R limits, the equations simplify due to rapidly decaying cutoff profiles $r(z)$. At first sight, it looks like all the traces on the right-hand side of the flow equation vanish and one is only left with (16), the equation of motion $E(R) = 0$. This is actually true on the hyperboloid, and the fixed-point solution is therefore the solution of $E(R) = 0$, namely,

我们现在以这些方程为例，详细说明如何构造渐近解。在大 R 极限下，由于截断轮廓 $r(z)$ 快速衰减，方程得到简化。乍看之下，流方程右侧的所有迹都消失了，仅留下运动方程 (16) $E(R) = 0$ 。这在双曲空间中实际成立，因此不动点解就是 $E(R) = 0$ 的解，即

$$f(R) = AR^2, \tag{48}$$

where A is an arbitrary constant. At any finite R , this is then accompanied by rapidly decaying corrections as discussed later, cf. Eq. (63).

其中 A 为任意常数。在任意有限 R 下，解都会带有后文讨论的快速衰减修正，参见式 (63)。

The story is different on the sphere since upon closer inspection, not all of the terms in the sums vanish. There are three such terms, the $n = 0$ and $n = 1$ components from $\mathcal{T}_0^{\hbar}$ and the $n = 1$ of $\mathcal{T}_0^{\text{Jac.}}$. To see this for the $n=0$ case, note that from Table 1, $\Delta_0 = -R/3$. Thus, using (27), the denominator of this term in the sum (20) is given by

> 球面上的情况则不同: 仔细观察后会发现，求和中的项并非全部消失。共有三个非零项: 来自 $\mathcal{T}_0^{\bar{h}}$ 的 $n=0$ 分量和 $n=1$ 分量，以及 $\mathcal{T}_0^{\text{Jac.}}$ 的 $n=1$ 分量。以 $n=0$ 的情况为例，由表 1 可得 $\Delta_0 = -R/3$ 。因此，利用式 (27)，求和式 (20) 中该项的分母为

$$9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16\mathcal{R}_k^{\bar{h}} \tag{49}$$

$$= R^2 f''(R) - Rf'(R) + E(R) + 16k^4 c_\phi r\left(\left[\alpha_0 - \frac{1}{3}\right]R\right)$$

Now, assuming that the leading asymptotic behavior is $f(R) = AR^2$, we see that the first two terms cancel each other, and likewise $E(R)$ vanishes, so we are left only with the cutoff term in the denominator. Therefore, this term takes the form of

> 现在假设主导渐近行为是 $f(R) = AR^2$ ，我们可以看到前两项相互抵消，同理 $E(R)$ 也消失，最终分母仅剩下截断项。因此，该项的形式为

$$\frac{1}{k^4 r(z)}\frac{d}{dt}\left[k^4 r(z)\right] = 4 - 2z\frac{d\ln r(z)}{dz} \tag{50}$$

with z set equal to $z = \left[\alpha_0 - \frac{1}{3}\right]R$.

> 其中 z 被设为等于 $z = \left[\alpha_0 - \frac{1}{3}\right]R$ 。

Turning to the $n=1$ components, note that from Table 1, both Δ_0 and Δ_1 vanish for $n=1$. In (20), apart from the cutoff term, the whole denominator therefore vanishes (because $E(R)$ vanishes). In (22), it is the second component that has a vanishing denominator apart from the cutoff term. The S_1 (first) component does not suffer from the same problem because there is also the $+R/3$ part in the denominator. However, the cutoff dependence is the same for the $n=1$ contributions, namely, $r(\alpha_0 R)$, and the numerical factors are such that these two $n=1$ contributions exactly cancel each other.

> 来看 $n=1$ 分量，根据表 1，当取 $n=1$ 时，Δ_0 和 Δ_1 都等于零。因此在式 (20) 中，除截断项外整个分母都为零 (因为 $E(R)$ 消失)。在式 (22) 中，第二个分量存在除截断项外分母为零的情况，而 S_1 (第一个) 分量不存在这个问题，因为其分母中还包含 $+R/3$ 项。不过，所有 $n=1$ 贡献的截断依赖都是相同的，即 $r(\alpha_0 R)$ ，并且数值因子恰好使得这两个 $n=1$ 贡献完全抵消。

Altogether then, effectively the only term on the RHS (right-hand side) of the flow equation that does not vanish asymptotically is the $n=0$ component of the $\mathcal{T}_0^{\bar{h}}$ trace. This is a problem however, since the $n=0$ component of $\mathcal{T}_0^{\bar{h}}$ contributes a term that grows at least as fast as R^2 . This is inconsistent with the fact that the LHS (left-hand side) of flow equation has been set to vanish asymptotically. Actually, this analysis shows that $f(R)$ grows faster than R^2 . For example, in the best-case scenario, the RHS $\sim R^2$, but that implies $f(R) \sim R^2 \ln R$ so that the LHS is left with an $E(R) \sim R^2$ to balance the contribution from the $n=0$ component of $\mathcal{T}_0^{\bar{h}}$.

因此，整体来看，流方程右侧渐近不消失的有效项只剩下 $\mathcal{T}_0^{\bar{h}}$ 迹的 $n = 0$ 分量。但这就带来了一个问题: $\mathcal{T}_0^{\bar{h}}$ 的 $n = 0$ 分量贡献了一个增长速度至少和 R^2 一样快的项，这和流方程左侧渐近为零的条件矛盾。实际上，该分析说明 $f(R)$ 比 R^2 增长更快。例如，在最优情况中，右侧为 $\sim R^2$，但这意味着 $f(R) \sim R^2 \ln R$，因此左侧需要保留一个 $E(R) \sim R^2$ 来抵消 $\mathcal{T}_0^{\bar{h}}$ 的 $n = 0$ 分量的贡献。

Therefore, we now assume that $f(R)$ actually grows faster than R^2 at large R. But this means we need to check again which terms in the traces have denominators that would vanish without a cutoff. By inspection, none of the traces that depend on $f(R)$ can now have this issue. In particular, the $n = 1$ component of the $\mathcal{T}_0^{\bar{h}}$ trace no longer has a denominator that could vanish, because $E(R)$ no longer vanishes at large R, while for the $n = 0$ component, the $f''(R)$ part in the denominator now dominates at large R. So the only contribution that survives on the RHS at large R is now the $n = 1S_2$ component of $\mathcal{T}_0^{\text{Jac}}$.

因此，我们现在假定 $f(R)$ 在大 R 处的增长速度确实快于 R^2。但这意味着我们需要再次检查迹中哪些项的分母会在无截断时变为零。经检查，依赖 $f(R)$ 的迹都不存在这个问题。具体而言，$\mathcal{T}_0^{\bar{h}}$ 迹的 $n = 1$ 分量不再存在可能变为零的分母，因为 $E(R)$ 在大 R 处不再为零；而对于 $n = 0$ 分量，分母中的 $f''(R)$ 项在大 R 处占主导。因此，大 R 下右侧仅存的贡献就是 $\mathcal{T}_0^{\text{Jac}}$ 的 $n = 1S_2$ 分量。

Keeping just this term, it turns out one can solve the fixed-point equation in closed form, thus obtaining the correct asymptotic behavior for general cutoff function $r(z)$. Using the values from Table 1, we have that the multiplicity of the $n = 1$ component is $D_{1,0} = 5$ and note that $m_{S_2} = 4$ and that $1/V = R^2/384\pi^2$ for the four-sphere. Thus, keeping only this leading term on the RHS of the flow equation, we have

仅保留这一项后，我们可以得到不动点方程的闭式解，从而得到一般截断函数 $r(z)$ 的正确渐近行为。利用表 1 中的数值，我们可知 $n = 1$ 分量的重数为 $D_{1,0} = 5$，且对于四维球面满足 $m_{S_2} = 4$ 和 $1/V = R^2/384\pi^2$。因此，仅保留流方程右侧的这一领头项，我们得到

$$2f(R) - Rf'(R) = \frac{R^2}{768\pi^2}\left[-10 + 5\alpha_0 R \frac{r'(\alpha_0 R)}{r(\alpha_0 R)}\right]. \tag{51}$$

This is exactly soluble. Indeed, dividing through by R^3, it can be rewritten as

该方程可直接求解。事实上，两边同除以 R^3 后，它可以改写为

$$-\frac{d}{dR}\left(\frac{f(R)}{R^2}\right) = \frac{1}{768\pi^2}\left[-\frac{10}{R} + 5\frac{d}{dR}\ln r(\alpha_0 R)\right], \tag{52}$$

which can be immediately integrated to give

对上式直接积分可得

$$f(R) = \frac{5R^2}{768\pi^2}\ln\frac{R^2}{r(\alpha_0 R)} + AR^2 + o\left(R^2\right) \text{ as } R \to +\infty, \tag{53}$$

where we included the integration constant A and finally we noted that terms that grow slower than R^2 will be generated by iterating this asymptotic solution to higher orders, hence the $o\left(R^2\right)$ part. The $\ln r$ term actually dominates, i.e., the large R behavior is dominated by cutoff-dependent effects. For example, using the cutoff (29) gives the first three terms in this series:

其中我们引入了积分常数 A ，并指出增长慢于 R^2 的项会通过将该渐近解迭代到更高阶产生，$o\left(R^2\right)$ 部分即来源于此。$\ln r$ 项实际占主导，也就是说大 R 行为由依赖截断的效应主导。例如，采用截断 (29) 可得到该级数的前三项:

$$f(R)=\frac{5a\alpha_0^b}{768\pi^2}R^{2+b}+\frac{5}{768\pi^2}R^2\ln R+AR^2$$

$$+\frac{16c_{\bar{h}}}{5ab(1+b)\alpha_0^b}\left(\alpha_0-\frac{1}{3}\right)\mathrm{e}^{-a\left(\alpha_0-\frac{1}{3}\right)^b R^b}+\cdots. \tag{54}$$

To get the next term in the series, the solution is substituted back into the fixed-point equation, and the next leading correction is isolated. This leads to the last displayed correction above. It is exponentially decaying and comes from the $n=0$ term in the $\mathcal{T}_0^{\bar{h}}$ trace. One finds that other corrections decay faster provided that $\alpha_0 < \frac{5}{6}+\alpha_1$. This is satisfied, thanks to the restrictions on the α_i parameters discussed in sections ”Sphere” and ”Hyperboloid.” Substituting (54) back into the fixed-point equation and proceeding similarly, one can in principle develop the whole asymptotic series. It is an infinite series of ever faster decaying terms and is indicated by the ellipses. In particular, these terms will include a power series in A .

为得到级数的下一项，需将解代回不动点方程，分离出下一个领头修正。上述最后一个展示的修正就是这样得到的。它是指数衰减的，来源于 $\mathcal{T}_0^{\bar{h}}$ 迹中的 $n=0$ 项。可以发现，只要满足 $\alpha_0<\frac{5}{6}+\alpha_1$ ，其他修正的衰减速度更快。由于“球面”节和“双曲面”节中已经讨论过对 α_i 参数的限制，该条件是满足的。将 (54) 代回不动点方程并按同样方法操作，原则上可以得到整个渐近级数。这是一个由衰减速度越来越快的项构成的无穷级数，省略号即表示这一点。具体而言，这些项将包含 A 的幂级数。

At this point, we have succeeded in finding consistent asymptotics. $f(R)$ does grow faster than R^2 on the sphere, as assumed, and using such a form in the RHS of the fixed-point equation, one can see that the $n=1S_2$ component of $\mathcal{T}_0{}^{\mathrm{Jac}}$ dominates at large R , which leads back to the above equation.

至此，我们已经成功找到了自洽的渐近行为。正如我们假设的那样，在球面上 $f(R)$ 的增长确实快于 R^2 ，将该形式代入不动点方程右侧可以发现，$\mathcal{T}_0{}^{\mathrm{Jac}}$ 的 $n=1S_2$ 分量在大 R 处占主导，由此即可推导出上述方程。

Recall that the fixed-point equation is actually second order. But the asymptotic solutions only have one free parameter A , even though there should be two. To find out where the second parameter has gone, we linearize about the fixed point $f(R)+\delta f(R)$ and plug it into the flow equation (47) to get

我们知道不动点方程实际是二阶方程，按理说应该有两个自由参数，但渐近解只有一个自由参数 A 。为了找出第二个参数的去向，我们在不动点 $f(R)+\delta f(R)$ 附近线性化，将其代入流方程 (47) 得到

$$-a_2(R)\,\delta f''(R)+a_1(R)\,\delta f'(R)+a_0(R)\,\delta f(R)=4\delta f(R), \tag{55}$$

with

其中

$$a_2 = \frac{144c_h}{V}\,\mathrm{Tr}\left[\frac{\Delta_0^2\left(2r\left(\Delta_0+\alpha_0 R\right)-\left(\Delta_0+\alpha_0 R\right)r'\left(\Delta_0+\alpha_0 R\right)\right)}{\left(9f''\left(R\right)\Delta_0^2+3f'\left(R\right)\Delta_0+E\left(R\right)+16c_h r\left(\Delta_0+\alpha_0 R\right)\right)^2}\right],$$

(56)

$$a_1 = 2R-\frac{16c_h}{V}\,\mathrm{Tr}\left[\frac{\left(3\Delta_0-R\right)\left(2r\left(\Delta_0+\alpha_0 R\right)-\left(\Delta_0+\alpha_0 R\right)r'\left(\Delta_0+\alpha_0 R\right)\right)}{\left(9f''\left(R\right)\Delta_0^2+3f'\left(R\right)\Delta_0+E\left(R\right)+16c_h r\left(\Delta_0+\alpha_0 R\right)\right)^2}\right]$$
$$+\frac{2c_T}{V}\,\mathrm{Tr}\left[\frac{\left(R/2-\Delta_2\right)\left(2r\left(\Delta_2+\alpha_2 R\right)-\left(\Delta_2+\alpha_2 R\right)r'\left(\Delta_2+\alpha_2 R\right)\right)}{\left(-f'\left(R\right)\Delta_2-E\left(R\right)/2+2c_T r\left(\Delta_2+\alpha_2 R\right)\right)^2}\right],$$

(57)

$$a_0 = \frac{32c_h}{V}\,\mathrm{Tr}\left[\frac{\left(2r\left(\Delta_0+\alpha_0 R\right)-\left(\Delta_0+\alpha_0 R\right)r'\left(\Delta_0+\alpha_0 R\right)\right)}{\left(9f''\left(R\right)\Delta_0^2+3f'\left(R\right)\Delta_0+E\left(R\right)+16c_h r\left(\Delta_0+\alpha_0 R\right)\right)^2}\right]$$
$$+\frac{2c_T}{V}\,\mathrm{Tr}\left[\frac{\left(2r\left(\Delta_2+\alpha_2 R\right)-\left(\Delta_2+\alpha_2 R\right)r'\left(\Delta_2+\alpha_2 R\right)\right)}{\left(-f'\left(R\right)\Delta_2-E\left(R\right)/2+2c_T r\left(\Delta_2+\alpha_2 R\right)\right)^2}\right]. \quad (58)$$

In the large R limit, $a_1(R) \sim 2R$, and a_0 and a_2 vanish asymptotically. Then it is tempting to simply set a_0 and a_2 to zero to find the asymptotic solution to (55). But if this is done, there is only one solution $\delta f(R) = \delta A R^2$. In fact, this is just the leading term in an asymptotic series which is nothing but what one would derive from (54) by differentiating with respect to A . (Recall that the ellipses actually contain a power series in A.) This asymptotic solution is an exact series solution to (55) where a_0 and a_2 are only involved in constructing the subleading corrections. To find more than the one parameter δA in the solution to (55), $\delta f''(R)$ cannot be neglected, implying that higher derivatives must dominate over lower ones in the large R limit. Hence, the other solution is one where $\delta f(R)$ can at first be neglected. Then writing (55) as

在大 R 极限下，$a_1(R) \sim 2R$ 、a_0 和 a_2 渐近趋近于零。因此我们很容易直接将 a_0 和 a_2 置零来求解 (55) 的渐近解，但这样做仅能得到一个解 $\delta f(R) = \delta A R^2$ 。实际上，这只是渐近级数中的首项，恰好就是对 (54) 关于 A 求导得到的结果。(请注意，省略号实际包含了关于 A 的幂级数。) 该渐近解是 (55) 的精确级数解，其中仅次导修正项的构造会涉及 a_0 和 a_2 。若要在 (55) 的解中得到单参数 δA 之外的更多结果，就不能忽略 $\delta f''(R)$ ，这意味着在大 R 极限下，高阶导数的贡献大于低阶导数。因此，另一种解可以先忽略 $\delta f(R)$ ，再将 (55) 改写为

$$\frac{d}{dR}\ln\delta f'(R) = \frac{a_1(R)}{a_2(R)} \;\Rightarrow \delta f(R) = B\int^R dR'\exp\int^{R'} dR''\frac{a_1(R'')}{a_2(R'')},$$

(59)

where B is the second parameter. For the explicit form, a_2 is needed. It gets its leading contribution from the same source as the last displayed term in (54). Using the same cutoff choice, (29), asymptotically

其中 B 是第二个参数，需要知道 a_2 才能得到显式形式。a_2 的首项贡献来源与 (54) 最后展示的项相同。采用相同的截断选择 (29)，渐近情况下

$$a_2(R) = \frac{24576\pi^2 c_{\bar{h}}}{25ab(1+b)^2\alpha_0^{2b}}\left(\alpha_0-\frac{1}{3}\right)^{1+b} R^{1-b}\mathrm{e}^{-a\left(\alpha_0-\frac{1}{3}\right)^b R^b}+\cdots. \quad (60)$$

Recalling that $a_1 = 2R$ to leading order, the integrals can be evaluated by successive integration by parts, as an asymptotic series and where each term is given in closed form.

由于 $a_1 = 2R$ 已经取到首阶，我们可以通过逐次分部积分计算积分，得到渐近级数，且每一项都有闭合形式。

Since this strategy is used many times in this kind of asymptotic analysis, let us sketch it on the indefinite integral

由于这类渐近分析中会多次用到该方法，我们以不定积分为例简要说明:

$$\int dRG(R)\,\mathrm{e}^{F(R)} = \frac{G(R)}{F'(R)}\mathrm{e}^{F(R)} - \int dR\left(\frac{G(R)}{F'(R)}\right)'\mathrm{e}^{F(R)}. \tag{61}$$

The above equality follows by integration by parts; however, if $F(R)$ grows at least as fast as R for large R , where F is either sign, and $G(R)$ grows or decays slower than an exponential of R , then the integral on the right is subleading compared to the integral on the left. Iterating this identity then evaluates the integral in the large R limit as $\mathrm{e}^{F(R)}$ times an asymptotic series, the first term on the RHS being the leading term.

上述等式由分部积分得到；但若大 R 下 $F(R)$ 增长速度不慢于 R ，其中 F 可正可负，且 $G(R)$ 增长或衰减的速度慢于 R 的指数函数，那么右侧积分相较于左侧积分是次导的。对该等式进行迭代，就可以得到大 R 极限下积分等于 $\mathrm{e}^{F(R)}$ 乘以一个渐近级数，右侧第一项就是首项。

In this way, using the cutoff (29), the solution (59) on the sphere turns out to be

通过这种方式，采用截断 (29)，球面上的解 (59) 最终为

$$\delta f(R) \sim B\exp\left\{\frac{12(1+b)^2\alpha_0^{2b}}{12288\pi^2 c_{\bar{h}}}\left(\alpha_0 - \frac{1}{3}\right)^{-1-2b} R\mathrm{e}^{a(\alpha_0-1/3)^b R^b}\right\}. \tag{62}$$

The analysis proceeds similarly on the hyperboloid [43]. As $R \to -\infty$, one finds

双曲面上的分析过程类似 [43]，当 $R \to -\infty$ 时，得到

$$f(R) = AR^2 + \frac{c_{S1}}{96\sqrt{3\pi a^3 b^3}}\left(\frac{25}{48} - \alpha_0\right)^{\frac{5-3b}{2}}(-R)^{2-\frac{3b}{2}} \tag{63}$$

$$\times\left\{1 + O\left(|R|^{-\frac{1}{2}}\right)\right\}\mathrm{e}^{-a\left[\left(\alpha_0-\frac{25}{48}\right)R\right]^b} + \cdots.$$

The correction is again a decaying exponential because α_0 is restricted to $\alpha_0 < 25/48$. All scalar traces (thus also A) contribute to the $O\left(|R|^{-\frac{1}{2}}\right)$ term, and the ellipses stand for terms with faster decaying exponentials. The asymptotic behavior of a_2 turns out now to be

由于 α_0 被限制在 $\alpha_0 < 25/48$ 范围内，修正项仍是衰减指数函数。所有标量迹 (因此也包括 A) 都对 $O\left(|R|^{-\frac{1}{2}}\right)$ 项有贡献，省略号代表衰减更快的指数项。此时 a_2 的渐近行为最终为

$$a_2(R) = \frac{4c_{\bar{h}}}{81A^2\sqrt{3\pi ab}}\left(\frac{25}{48} - \alpha_0\right)^{\frac{5-b}{2}}(-R)^{1-\frac{b}{2}}\mathrm{e}^{-a\left[\left(\alpha_0-\frac{25}{48}\right)R\right]^b} + \cdots \tag{64}$$

(the ellipses being faster decaying terms). And thus one finds on the hyperboloid

(省略号为衰减更快的项)。因此我们可以得到双曲面上的结果:

$$\delta f(R) \sim B \exp\left\{\frac{81A^2}{2c_{\bar{h}}}\sqrt{\frac{3\pi}{ab}}\left(\frac{25}{48}-\alpha_0\right)^{-\frac{b+5}{2}}(-R)^{1-b/2}\mathrm{e}^{a[(\alpha_0-25/48)R]^b}\right\}.$$

(65)

However, there is a problem here. Both these solutions (62) and (65) for $\delta f(R)$ are rapidly growing exponentials of an exponential. In the asymptotic regime, these perturbations are no longer small, thus invalidating the initial linearization assumption used to derive them. Therefore, these solutions must be discarded, and thus, we conclude that the fixed points have only one free parameter on both the sphere and hyperboloid.

但这里存在一个问题: 针对 $\delta f(R)$ 的这两个解 (62) 和 (65) 都是指数上指数的快速增长函数。在渐近区域中，这些扰动不再是小量，因此推导时使用的初始线性化假设不成立。这些解必须被舍弃，由此我们得出结论: 球面和双曲面上的不动点都仅有一个自由参数。

These results allow us to draw important conclusions. Each of the asymptotic fixed-point solutions, (54) and (63), contributes one constraint on the flow equation (For example, at some initial very large R , we can set $2f(R) = Rf'(R)$ since this boundary condition imposes the leading behavior (48). Using the subleading corrections, we can furnish a more accurate Robin boundary condition at more reasonable values of R .). There are no boundary conditions coming from $R = 0$, so we can expect one-parameter sets of fixed-point solutions on both $\mathbb{S}^4$ and $\mathbb{H}^4$. At first sight, this is a disappointing result for the asymptotic safety program. However, if we now use $\mathbb{R}^4$, Eq. (46), to match smoothly between these solutions, then we have two boundary conditions on a second-order differential equation, one coming from the sphere and one from the hyperboloid. Thus, there can now only be at most a discrete set of solutions. In the next section, more evidence will be presented for why these topologies should be considered smoothly joined together in this way.

这些结果让我们能够得出重要结论。渐近不动点解 (54) 和 (63) 中的每一个都对流方程给出了一个约束 (例如，在某个初始极大值 R 处，我们可以设定 $2f(R) = Rf'(R)$ ，因为该边界条件给出了主阶行为 (48)。利用次领头阶修正，我们可以在更合理的 R 取值下得到更精确的罗宾边界条件)。$R = 0$ 不提供任何边界条件，因此我们可以预期在 $\mathbb{S}^4$ 和 $\mathbb{H}^4$ 上都存在单参数族的不动点解。乍看之下，这对渐近安全程序来说是一个令人失望的结果。但如果我们现在利用 $\mathbb{R}^4$ 即式 (46)，让这些解之间平滑匹配，那么我们就得到了二阶微分方程的两个边界条件，一个来自球面，一个来自双曲面。因此，解最多只能是离散集合。下一节将给出更多证据，说明为什么应当认为这些拓扑可以通过这种方式平滑连接。

Numerical Solution

数值解

Note that this does not answer the question of whether there is more than one fixed point, or no fixed point at all, or the phenomenologically preferred answer: a single fixed point. As already mentioned, the only way to find out which of these latter possibilities is realized is to perform an extensive numerical search for such global $f(R)$ solutions. As we have just seen, the asymptotic fixed-point solutions (54) and (63), on the sphere and hyperboloid, respectively, depend on one single parameter, which we called A on both sides. Even if there is a global solution that connects them, the value of A will almost surely be different in the two different topologies. On one of the topologies, one can scan through A at some large value of R and solve the equations backward toward $R=0$. With the exponential cutoff profile (29), solutions have been found on the sphere this way, in a narrow region around $A=-0.01$, matching to the asymptotic series at $R\sim 10$. On the hyperboloid, no solution was found however, although a more comprehensive numerical analysis might find one [43].

请注意，这并未解答不动点的数量问题: 是否存在多个不动点、根本不存在不动点，或是唯象学更偏好的单个不动点。如前所述，要确定究竟是哪一种情况，唯一的方法就是对这类全局 $f(R)$ 解展开大规模数值搜索。正如我们所见，分别定义在球面和双曲面上的渐近不动点解 (54) 与 (63) 都仅依赖一个单参数，在两种情形下我们都将该参数记为 A 。即使存在连接二者的全局解，两种不同拓扑结构下 A 的值也几乎必然不同。对于其中一种拓扑，我们可以在 R 的较大取值范围内扫描 A ，然后向 $R=0$ 方向逆向求解方程。采用指数截断轮廓 (29) 时，人们已经通过这种方法在球面上找到了解，解存在于 $A=-0.01$ 附近的狭窄区域，且与 $R\sim 10$ 处的渐近级数匹配。然而在双曲面上尚未找到解，但更全面的数值分析或许能发现解的存在 [43]。

Eigenoperators

本征算子

So far, we have analyzed the flow equation in the $f(R)$ approximation at the fixed point (where $\partial_t f(R) = 0$). Assuming that there is a global solution, we now turn to the question of whether the theory is predictive. This is answered by solving the eigenvalue equation and figuring out how many relevant operators the fixed-point solution has. Relevant operators are the ones that fall into the fixed point when increasing the cutoff scale k . The number of these operators corresponds to the number of parameters that will have to be fixed experimentally. We now prove that in this second-order formulation, if we take the equations to apply simultaneously across the three spaces $\mathbb{S}^4, \mathbb{R}^4$, and $\mathbb{H}^4$, there are a finite number of relevant operators

到目前为止，我们已经分析了不动点处 $f(R)$ 近似下的流方程 (其中满足 $\partial_t f(R)=0$)。假设存在全局解，我们现在转向该理论是否具有预言性的问题。这个问题可以通过求解本征值方程并确定不动点解包含多少相关算子来解答。相关算子是当截断标度 k 增大时向不动点演化的算子。这些算子的数量对应需要通过实验确定的参数数量。我们现在证明，在这个二阶表述中，如果假设方程同时适用于三个空间 $\mathbb{S}^4, \mathbb{R}^4$ 和 $\mathbb{H}^4$ ，则相关算子的数量是有限的

[42]. Plugging (5) into the flow equation (18), we get a second-order ordinary differential eigenvalue equation:

[42]。将式 (5) 代入流方程 (18)，我们得到二阶常微分本征值方程:

$$-a_2(R)\,v''(R) + a_1(R)\,v'(R) + a_0(R)\,v(R) = \lambda v(R) \tag{66}$$

where the eigenvalues $\lambda = 4-\theta, v(R)$ is the eigenoperator and the a_i s are given by Eqs. (56-58).

其中本征值 $\lambda = 4 - \theta, v(R)$ 为本征算子，且 a_i s 由式 (56-58) 给出。

Asymptotic Analysis

渐近分析

The first step is to apply asymptotic analysis to the eigenoperator equation. The procedure closely follows that for the fixed point in section ”Asymptotic Analysis.” As already noted there, a_0 and a_2 decay exponentially fast, and in the large R limit, $a_1 \sim 2R$. Then the asymptotic form of the eigenvalue equation is

第一步是对本征算符方程进行渐近分析。步骤与“渐近分析”一节中不动点的处理方法高度一致。正如该节已经指出的，a_0 和 a_2 呈指数衰减，在大 R 极限下，$a_1 \sim 2R$ 。因此本征值方程的渐近形式为

$$\lambda v(R) - 2Rv'(R) = -a_2(R)\, v''(R). \tag{67}$$

Starting with the left-hand side, the solution is

从左侧开始，解为

$$v(R) \propto |R|^{\frac{\lambda}{2}} + \cdots, \tag{68}$$

where the ellipses stand for subleading corrections from the a_i s , in particular from the RHS. The solutions have one parameter, the constant of proportionality. The missing parameter must come from a solution for which $v''(R)$ cannot be neglected. But this implies diverging derivatives, and thus, $v(R)$ can be neglected. The equation is then analogous to what we had before where the second solution is now $v(R) \sim \delta f(R)$ in (62) on the sphere and (65) on the hyperboloid.

其中椭圆表示来自 a_i s 的次领头修正，尤其是来自方程右侧的修正。解只有一个参数，即比例常数。缺失的参数必须来自不能忽略 $v''(R)$ 的解。但这意味着导数发散，因此可以忽略 $v(R)$ 。该方程与我们之前得到的结果类似: 第二个解就是球面情形式 (62) 和双曲面情形式 (65) 中的 $v(R) \sim \delta f(R)$ 。

Now we ask whether these solutions are actually valid. The linearized solution (5) is meant to describe the RG flow ”close” to the fixed point. For any fixed ε , if $|v(R)/f(R)| \to \infty$ as $R \to \pm\infty$, that is not necessarily true since linearization is no longer valid. In this case, one can set

现在我们要确认这些解是否实际有效。线性化解 (5) 用于描述“靠近”不动点的重整化群流。对任意固定的 ε ，如果当 $R \to \pm\infty$ 时 $|v(R)/f(R)| \to \infty$ ，线性化就不再成立，结论不一定正确。这种情况下可以设

$$f_k(R) = f(R) + \varepsilon v_k(R) \tag{69}$$

and, without linearizing, ask for the correct evolution for $v_k(R)$ at large R . For large negative R , the RHS of the flow equation (18) can be neglected. For large positive R , the RHS of the flow equation can be neglected except for the $n = 1S_2$ component of $\mathcal{T}_0^{\mathrm{Jac}}$, which however just cancels the contributions from the LHS that grow faster than R^2 resulting from $f(R)$, cf. (54). Since in fact the $O(R^2)$ part of $f(R)$ also vanishes from the LHS (on both sphere and hyperboloid), in the large R regime, one has

不进行线性化，直接求解大 R 下 $v_k(R)$ 的正确演化。对于大的负 R，可以忽略流方程 (18) 的右侧。对于大的正 R，除了 $\mathcal{T}_0^{\mathrm{Jac}}$ 的 $n = 1S_2$ 分量外，流方程的右侧都可以忽略；而该分量恰好抵消了左侧中来自 $f(R)$、增长快于 R^2 的贡献，参见式 (54)。实际上由于 $f(R)$ 的 $O(R^2)$ 部分在左侧也会消失 (球面和双曲面都是如此)，因此在大 R 区域有

$$\partial_t v_k(R) - 2Rv_k'(R) + 4v_k(R) = o\left(R^2\right). \tag{70}$$

Any part of $v_k(R)$ growing at least as fast as R^2 is then easily solved for and gives mean-field evolution involving some arbitrary function v:

任何增长速度不慢于 R^2 的 $v_k(R)$ 分量都可以轻易求解，得到包含任意函数 v 的平均场演化:

$$v_k(R) = \mathrm{e}^{-4t} v\left(R\mathrm{e}^{2t}\right) + o\left(R^2\right). \tag{71}$$

It will be the same function v that was introduced in the linearized solution (5) if one requires as boundary condition $v_k(R) = v(R)$ at $k = \mu$. The question that remains is whether the RG evolution (71) is consistent with what we found by linearizing.

如果将 $v_k(R) = v(R)$ 在 $k = \mu$ 处的结果作为边界条件，得到的就是线性化解 (5) 中引入的同一个函数 v。剩下的问题是重整化群演化 (71) 是否与我们线性化得到的结果一致。

For the power law solution (68), linearization is valid at large $|R|$ if and only if $\lambda \leq 4$. This follows from the hyperboloid fixed-point asymptotics (63), the sphere side (54) requiring only the weaker constraint, $\lambda \leq 4 + 2b$. On the other hand, if $\lambda > 4$, one can use the general perturbation (69), finding the solution (71). Substituting the explicit form (68) of the boundary condition gives

对于幂律解 (68)，当且仅当 $\lambda \leq 4$ 时，线性化在大 $|R|$ 下成立。这可以由双曲面不动点的渐近行为 (63) 得到，球面一侧 (54) 只要求更弱的约束 $\lambda \leq 4 + 2b$。另一方面，如果 $\lambda > 4$，可以使用一般微扰 (69) 得到解 (71)。代入边界条件的显式形式 (68) 可得

$$v_k(R) = v(R)\,\mathrm{e}^{-\theta t} + o\left(R^2\right), \tag{72}$$

where $\theta = 4 - \lambda$, i.e., the linearized solution (5) is reproduced. We conclude that asymptotically, power law eigenoperators (68) are valid solutions for any λ. Their t evolution is multiplicative and given by the flow of a conjugate coupling $g(t) = \varepsilon \mathrm{e}^{-\theta t}$, cf. (5).

其中 $\theta = 4 - \lambda$，即线性化解 (5) 得以重现。我们得出结论: 渐近来看，幂律本征算符 (68) 对任意 λ 都是有效解。它们的 t 演化是乘性的，由共轭耦合 $g(t) = \varepsilon \mathrm{e}^{-\theta t}$ 的流给出，参见式 (5)。

On the other hand, the solutions that behave asymptotically as $v(R) \sim \delta f(R)$ are growing exponentials of exponentials. Linearization is not valid at large $|R|$, where the t dependence is given instead by (71). Now we cannot separate out the t dependence. Therefore, such perturbations cannot be regarded as eigenoperators evolving multiplicatively.

另一方面，渐近行为满足 $v(R) \sim \delta f(R)$ 的解是指数的指数增长型。大 $|R|$ 处线性化不再成立，此时 t 的依赖关系由式 (71) 给出。我们现在无法分离出 t 依赖，因此这类微扰不能被看作按乘法演化的本征算符。

Excluding them leads to quantization of the spectrum. This is because the large R dependence (68) provides a boundary condition on both the sphere and the hyperboloid side and linearity provides a further boundary condition since one can choose a normalization, e.g., $v(0) = 1$. These three conditions over-constrain the eigenoperator equation (66) leading to the quantization of λ , i.e., to a discrete eigenoperator spectrum.

将这类解排除后会得到谱量子化。这是因为大 R 处的依赖关系 (68) 在球面和双曲面侧都给出了边界条件，而线性性质给出了额外边界条件 (我们可以选取归一化，例如 $v(0) = 1$)。这三个条件对本征算符方程 (66) 给出了过约束，导致 λ 量子化，即得到离散的本征算符谱。

Sturm-Liouville Theory

施图姆-刘维尔理论

Sturm-Liouville-type equations take the form

施图姆-刘维尔型方程具有如下形式

$$Lv(R) = \lambda w(R) v(R), \tag{73}$$

where L is the self-adjoint operator

其中 L 是自伴算子

$$L = -\frac{d}{dR}\left(p(R)\frac{d}{dR}\cdot\right) + q(R), \tag{74}$$

with $p(R)$ and $q(R)$ being real functions and $w(R)$ also being positive. For the second-order formulation, the eigenvalue equation can be put in this form. The properties of these equations will then allow us to draw conclusions about the spectrum of the eigenvalues.

其中 $p(R)$ 和 $q(R)$ 为实函数，且 $w(R)$ 恒正。对于二阶表述形式，特征值方程可以整理为该形式。这类方程的性质将帮助我们推导得出特征值谱的相关结论。

The weight function is defined as

权函数定义为

$$w(R) = \frac{1}{a_2(R)} \exp - \int^R dR' \frac{a_1(R')}{a_2(R')}, \tag{75}$$

since multiplying with the eigenvalue equation (66) and rearranging cast it in Sturm-Liouville form:

这是因为将其乘到特征值方程 (66) 上整理后，即可将方程化为施图姆-刘维尔形式:

$$-(a_2(R) w(R) v'(R))' + w(R) a_0(R) v(R) = \lambda w(R) v(R). \tag{76}$$

Notice that the trace in a_2 is positive. This is because the cutoff is monotonically decreasing; hence, $r'(z) < 0$ and $r(z) > 0$, so the sign of a_2 depends on $c_{\bar{h}}$, which is positive. This implies that the weight function $w(R) > 0$ as required.

注意 a_2 中的迹为正。这是因为截断函数是单调递减的；因此 $r'(z) < 0$ 和 $r(z) > 0$ ，故 a_2 的符号由 $c_{\bar{h}}$ 决定，而 $c_{\bar{h}}$ 为正。这说明权函数 $w(R) > 0$ 满足要求。

Next we check if the operator is self-adjoint. Taking $v = v_j(R)$, multiplying by $v_i(R)$, and integrating over R give

接下来我们验证该算子是否为自伴算子。取 $v = v_j(R)$ ，乘以 $v_i(R)$ 并对 R 积分可得

$$-\int v_i L v_j = -\int v_i \left(a_2 w v_j'\right)' + \int v_i a_0 w v_j. \tag{77}$$

If L is self-adjoint, then this should be the same for $j \leftrightarrow i$. The first term on the RHS can be written as

若 L 是自伴算子，则对 $j \leftrightarrow i$ 该等式也应成立。右侧第一项可以写为

$$-\int \left[v_i \left(a_2 w v_j'\right)\right]' + \int \left[v_j \left(a_2 w v_i'\right)\right]' - \int \left(a_2 w v_i'\right)' v_j. \tag{78}$$

Thus, what is required is that the first two terms above cancel each other. This is automatically satisfied if R is taken to have the full range since $w(R) \to 0$ exponentially fast as $R \to \pm\infty$. If the differential equation is restricted to either the four-sphere or four-hyperboloid, there would be a boundary at $R = 0$. The weight function does not vanish there, and thus, the operator L would then not be selfadjoint. This is another powerful hint that the correct treatment is to smoothly join the three topologies together. Note also that none of these equations would make sense if the exponentially growing set of solutions $v(R) \sim \delta f(R)$ are included, where $\delta f(R)$ is given by (62) or (65). From (59) and (61), one can see that actually these $\delta f(R) \sim 1/w(R)$ and thus such $v(R)$ are not square integrable under the weight function $w(R)$ since $w(R) v^2(R) \sim 1/w(R)$, which diverges at large R . Hence, this condition only picks out the correct solutions from the eigenvalue equation and justifies the use of Sturm-Liouville techniques.

因此，我们需要上述前两项相互抵消。若 R 取全定义域，该条件自动满足，因为当 $R \to \pm\infty$ 趋于无穷时，$w(R) \to 0$ 会指数衰减。若微分方程仅限制在四维球面或四维双曲面上，则会在 $R = 0$ 处存在边界。权函数在边界处不为零，因此算子 L 不是自伴算子。这是一个强有力的提示，说明正确的处理方式是将三种拓扑光滑连接。另外还需注意，若包含指数增长的解族 $v(R) \sim \delta f(R)$ (其中 $\delta f(R)$ 由 (62) 或 (65) 给出)，这些方程都不成立。从 (59) 和 (61) 可以看出，实际上这些 $\delta f(R) \sim 1/w(R)$ 因此这类 $v(R)$ 在权函数 $w(R)$ 下不是平方可积的，因为 $w(R)v^2(R) \sim 1/w(R)$ 在 R 很大时发散。因此，该条件只会从特征值方程中筛选出正确的解，为使用施图姆-刘维尔方法提供了依据。

Thus, when restricted to perturbations that grow only as a power at large $|R|$, the eigenvalue equation (66) is of Sturm-Liouville type. The consequences for the spectrum of the eigenvalues can be seen by a standard transformation to Liouville normal form. Define a coordinate x as

因此，当限制在大 $|R|$ 处仅按幂次增长的微扰时，特征值方程 (66) 属于施图姆-刘维尔型。我们可以通过标准变换得到刘维尔标准型，进而得到特征值谱的结论。定义坐标 x 为

$$x = \int_0^R \frac{1}{\sqrt{a_2(R')}} dR'. \tag{79}$$

Then $x \to \pm\infty$ as $R \to \pm\infty$ because $a_2(R)$ vanishes at large $|R|$. Defining the "wave function"

此时 $x \to \pm\infty$ 趋近于 $R \to \pm\infty$ ，因为 $a_2(R)$ 在大 $|R|$ 处趋于零。定义“波函数”

$$\psi(x) = a_2^{\frac{1}{4}}(R)\, w^{\frac{1}{2}}(R)\, v(R), \tag{80}$$

(66) can be transformed into

(66) 可以变换为

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = \lambda\psi(x), \tag{81}$$

which is just the one-dimensional Schrödinger equation with energy λ. The potential turns out to be [42]

这就是一维薛定谔方程，其能量为 λ。势能已被证明为 [42]

$$U(x) = a_0 + \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2'\left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\right) - \frac{a_2''}{4}. \tag{82}$$

This potential has no singularities at finite x. Asymptotically, the term proportional to a_1^2 will dominate for $x \to \pm\infty$ and thus the potential $U(x) \to +\infty$. This then implies the following important properties:

该势能在有限 x 处没有奇点。渐近上，当 $x \to \pm\infty$ 时，与 a_1^2 成正比的项占主导，因此势能 $U(x) \to +\infty$ 。由此可得到下述重要性质:

(1) The eigenvalues λ_n are discrete, real, and non-degenerate.

(1) 特征值 λ_n 是离散、实数且非简并的。

(2) There exists a lowest eigenvalue λ_0 (i.e., bounded from below).

(2) 存在最小本征值 λ_0 (即下方有界。

(3) The only accumulation point is at infinity.

(3) 唯一聚点位于无穷远。

Asymptotic analysis already showed that the eigenvalues are discrete, but this Sturm-Liouville analysis allows to conclude much more. Now it is straightforward to see that there are a finite number of relevant operators such that $\theta_n = 4 - \lambda_n \geq 0$. Indeed, this is so because $\lambda_n \to \infty$ as $n \to \infty$ and because there exists a lowest eigenvalue λ_0 .

渐近分析已经表明本征值是离散的，但施图姆-刘维尔分析可以得到更多结论。现在可以直接看出，满足 $\theta_n = 4 - \lambda_n \geq 0$ 的 relevant 算符数量是有限的。事实上，这是因为 $\lambda_n \to \infty$ 当 $n \to \infty$ ，同时存在最小本征值 λ_0 。

But these results should be accepted with caution. Recall that to obtain them, some severe approximations were used, such as the single-metric approximation and the truncation to the function $f(R)$. One way to judge the validity of the results is to check the extent to which they are scheme independent (universal), in particular independent of the choice of cutoff. It turns out that the critical exponents θ_n can be solved for analytically, again by using asymptotic analysis, and this gives a precise way to answer the question of scheme dependence in this regime.

但这些结果需要谨慎看待。需要注意，为得到这些结果我们使用了一些强近似，例如单度规近似和对函数 $f(R)$ 的截断。判断结果有效性的一种方法是检验它们在多大程度上是方案无关的 (普适的)，尤其是否与截断选择无关。事实表明，临界指数 θ_n 同样可以通过渐近分析解析求解，这为回答该区域的方案依赖问题提供了精确方法。

From (60) and (64), the reader can see that the leading contribution to a_2 takes the following form on both sphere and hyperboloid:

由 (60) 和 (64)，读者可以看到，对 a_2 的领头贡献在球面和双曲面上都具有如下形式:

$$a_2(R) = \frac{1}{G^2(R)} \mathrm{e}^{-2F(R)} \tag{83}$$

where $F(R)$ is positive and proportional to $|R|^b$ and $G(R)$ goes like a power of R . They therefore satisfy the conditions required to use the trick (61) on the equation (79) defining x . Then asymptotically

其中 $F(R)$ 为正，且正比于 $|R|^b$ ，$G(R)$ 是 R 的幂次。因此它们满足对定义 x 的方程 (79) 使用技巧 (61) 所需的条件。渐近形式如下

$$x = \frac{G(R)}{F'(R)} \mathrm{e}^{F(R)} + \ldots \tag{84}$$

where the ellipses stand for multiplicative subleading terms. Alternatively, this can be seen by differentiating (79) and (84) with respect to R . The potential can then be approximated to leading order as

其中省略号代表乘性次领头项。或者，也可以通过对 R 微分 (79) 和 (84) 得到该结果。势可以领头阶近似为

$$U(x) = \frac{a_1^2}{4a_2} = \frac{R^2}{a_2(R)} = [RF'(R)]^2 x^2. \tag{85}$$

Evidently, $RF'(R) = bF(R)$, and thus, taking logs of (84),

显然，$RF'(R) = bF(R)$ ，因此对 (84) 取对数得

$$U(x) = (bx \ln|x|)^2 \left\{1 + O\left(\frac{\ln\ln|x|}{\ln|x|}\right)\right\} \quad x \to \pm\infty, \tag{86}$$

where in the equation above the order of the subleading correction is also indicated. (The latter requires taking into account iterations of (61) and the subleading corrections to a_2 .) Using the WKB approximation, one can then find the critical exponents for large n [43]:

上述方程中同时标出了次领头修正的阶数。(后者需要考虑 (61) 的迭代和对 a_2 的次领头修正。) 利用 WKB 近似，就可以得到大 n 时的临界指数 [43]:

$$\theta_n = -b(n \ln n)\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\} \quad \text{as} \quad n \to \infty. \tag{87}$$

The result shows almost a linear dependence on n . This much is similar to extensive numerical work done on large polynomial truncations of a third-order formulation up to $n \leq 70$ [17]. These authors find near-Gaussian scaling dimension. They use an adaptive cutoff so there is no direct comparison, and they use the optimized profile (26) with no free parameters in the cutoff, so universality is not tested in this way. Indeed, the scaling dimension should be universal. The leading behavior of this expression is independent of all parameters in the chosen general family of cutoffs, except one, namely, the parameter b in (29). Explicitly, it is independent of a in (29) and of all the c_ϕ and α_i . Unfortunately, the dependence on b still amounts to strong dependence.

结果显示几乎对 n 呈线性依赖。这和已有大量针对三阶表述大多项式截断 (最高到 $n \leq 70$) 的数值工作结果相似 [17]。这些作者得到了近高斯标度维数。他们使用自适应截断，因此无法直接比较，且他们在截断中使用优化轮廓 (26) 且无自由参数，因此没有以这种方式检验普适性。标度维数本应是普适的。该表达式的领头行为与所选一般截断族中的所有参数无关，仅与一个参数有关，即 (29) 中的参数 b 。具体来说，它与 (29) 中的 a 以及所有 c_ϕ 和 α_i 都无关。遗憾的是，对 b 的依赖仍然是强依赖。

Actually, this remaining dependence is an artifact of the single-metric approximation [2] (More generally, single-field approximations are a known source of artifacts [56].). We have seen that it comes from the R^b dependence of $F(R)$ in (83), equivalently (60) and (64). This in turn arises from the cutoff dependence in

Eq. (56) and in particular the cutoff profile’s dependence on R (through in fact the lowest eigenvalue). To see that the dependence in (87) is an artifact of the single-metric approximation, imagine for the moment that the single-metric approximation was not made and yet somehow the initial ansatz (2) still made sense. (In reality, such a simple ansatz would no longer be possible because diffeomorphism invariance is replaced by BRST invariance for the quantum fields and furthermore it is badly broken, but let us overlook that for the moment.) Now the curvature in it is the full quantum curvature $\hat{R}$, due to the full quantum metric $\hat{g}_{\mu\nu}$ in (6). The trace and the cutoff in (56) come from summing over modes on the background manifold in (3) so they depend on the background curvature R. The Hessian in (3) will result in differentiating $f\left(\hat{R}\right)$ with respect to the fluctuation field $h_{\mu\nu}$ or equivalently differentiating with respect to $\hat{g}_{\mu\nu}$. Thus, ultimately the eigenoperator perturbation equation (55) would take the form

> 实际上，这种剩余依赖是单度量近似 [2] 的人为产物 (更一般地说，单场近似是已知的人为产物来源 [56])。我们已经看到，它来自 (83)(等价于 (60) 和 (64)) 中 $F(R)$ 对 R^b 的依赖。而这又源自式 (56) 中的截断依赖，尤其是截断轮廓对 R 的依赖 (实际上是通过最低本征值产生的)。为了说明 (87) 中的这种依赖是单度量近似的人为产物，暂且假设我们没有采用单度量近似，但初始假设 (2) 仍然在某种程度上成立。(实际上，如此简单的假设不再可行，因为对于量子场，微分同胚不变性被 BRST 不变性取代，而且它还被严重破坏，但我们暂且忽略这一点。) 此时，其中的曲率是由 (6) 中的全量子度量 $\hat{g}_{\mu\nu}$ 给出的全量子曲率 $\hat{R}$ 。(56) 中的迹和截断来自对 (3) 中背景流形上的模式求和，因此它们依赖于背景曲率 R 。(3) 中的黑塞矩阵会对涨落场 $h_{\mu\nu}$ 求导 $f\left(\hat{R}\right)$，或者等价地对 $\hat{g}_{\mu\nu}$ 求导。因此，最终本征算符微扰方程 (55) 会形如

$$-a_2\left(R,\hat{R}\right)\delta f''\left(\hat{R}\right)+a_1\left(R,\hat{R}\right)\delta f'\left(\hat{R}\right)+a_0\left(R,\hat{R}\right)\delta f\left(\hat{R}\right)=4\delta f\left(\hat{R}\right) \tag{88}$$

with in particular

> 尤其满足

$$a_2=\frac{144c_h}{V}\,\mathrm{Tr}\left[\frac{\Delta_0^2\left(2r\left(\Delta_0+\alpha_0 R\right)-\left(\Delta_0+\alpha_0 R\right)r'\left(\Delta_0+\alpha_0 R\right)\right)}{\left(9f''\left(\hat{R}\right)\Delta_0^2+3f'\left(\hat{R}\right)\Delta_0+E\left(\hat{R}\right)+16c_h r\left(\Delta_0+\alpha_0 R\right)\right)^2}\right].$$

(89)

In deriving (87), one is interested in the large $\hat{R}$ dependence of (88). This depends on the large $\hat{R}$ dependence of the fixed-point functional $f\left(\hat{R}\right)$, and this feeds in to the coefficients $a_i\left(R,\hat{R}\right)$. But there is no $\exp\left(-a\hat{R}^b\right)$ dependence because the cutoff profile r depends only on the background curvature R, either directly or through the Laplacians whose eigenvalues only depend on the background manifold.

> 在推导 (87) 时，我们关注的是 (88) 对大 $\hat{R}$ 的依赖。这依赖于不动点泛函 $f\left(\hat{R}\right)$ 对大 $\hat{R}$ 的依赖，进而传递给系数 $a_i\left(R,\hat{R}\right)$。但其中不存在 $\exp\left(-a\hat{R}^b\right)$ 依赖，因为截断轮廓 r 仅依赖于背景曲率 R ——无论是直接依赖还是通过拉普拉斯算子依赖，而拉普拉斯算子的本征值仅依赖于背景流形。

Cross-References

> ## 交叉引用

Asymptotic Safety and Cosmology

渐近安全与宇宙学

Asymptotic Safety of Gravity with Matter

含物质引力的渐近安全

Black Holes in Asymptotically Safe Gravity

渐近安全引力中的黑洞

Form Factors in Asymptotically Safe Quantum Gravity

渐近安全量子引力中的形状因子

The Functional Renormalization Group in Quantum Gravity

量子引力中的泛函重整化群

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